POLYGONAL NUMBERS.

384. Triangular numbers are the successive sums of arithmetical series, whose first term is unity and common difference is also unity.

Hence the general triangular number may be expressed by

\[ 1 + 2 + 3 + \ldots + n = (1 + n) \frac{n}{2} \]

Now \( (1 + n) \frac{n}{2} \times 8 + 1 = 4n^2 + 4n + 1 = (2n + 1)^2 \)

which is a square number, \( \therefore \&c. \ldots \)

385. Hexagonal numbers are the successive sums of arithmetical series, whose first term is unity and common difference is \( 4 \).

\[ \therefore \text{the } n^{\text{th}} \text{ term in the series of hexagonal numbers} = 1 + 5 + 9 + \ldots + 1 + 4(n - 1) = 2 + 4(n - 1). \frac{n}{2} = 4 \frac{n(n - 2)}{2} = \frac{n(n - 2)}{2} = n (2n - 1) \]

The \((2n - 1)^{\text{th}}\) term in the series of triangular numbers

\[ = 1 + 2 + 3 + \ldots + (2n - 1) = (1 + 2n - 1) \frac{2n - 1}{2} = \frac{2n}{2} \times (2n - 1) \]

\[ = n (2n - 1) \]

\[ \therefore \text{the } n^{\text{th}} \text{ term, } &c. \ldots &c. \]
PRIME NUMBERS.

386. If \((m)\) be integral, every coefficient in the expansion of \((a + b)^m\) is integral (Barlow, p. 17). The general coefficient is of the form \(m^\frac{m-1}{2} \frac{m-2}{3} \ldots\). Hence when \(m\) is prime, \(m^\frac{m-1}{2} \frac{m-2}{3} \ldots\) must be integral; i.e., every coefficient except the first and last, is divisible by \(m\), and \(\therefore\) the sum of the intermediate terms is divisible by \(m\).

Now \(x^m = (x - 1 + 1)^m = (x - 1)^m + 1 + m \cdot Q_1\), \(Q_1\) being the sum of all the terms except the first and last, divided by \(m\).

Similarly \(x^2 = (x - 2 + 2)^m = (x - 2)^m + 1 + m \cdot Q_2\), \(Q_2 = \&c. = \&c.\).

\[
(x - x - 1)^m = (x - x)^m + 1 + m \cdot Q_x
\]

\[
\therefore x^m = 1 + 1 \ldots \ldots x \text{ terms} + m \cdot (Q_1 + Q_2 + \ldots \ldots Q_x)
\]

\[
= x + m \cdot (Q_1 + Q_2 + \ldots \ldots Q_x)
\]

\[
\therefore \frac{x^m}{m} = \frac{x}{m} + Q_1 + Q_2 + \ldots \ldots Q_x
\]

\[
\therefore \frac{x^m}{m} \text{ leaves the same remainder as} \frac{x}{m}
\]

Similarly if \(m\) be any prime \(> 2\), \(\frac{x^m}{2m}, \frac{x}{2m}\) leave the same remainders; also if \(m\) be any prime \(> 2 \times 3\), \(\frac{x^m}{2 \times 3m}, \frac{x}{2 \times 3m}\) leave the same remainders, \&c. \&c.

Again \(\frac{n^p}{p} = \frac{n^{p-1}}{p} \times n = \left(\frac{w + \frac{1}{p}}{p}\right) \frac{n}{p} = nw + \frac{n}{p}\)
PRIME NUMBERS.

But \( \frac{a^n}{p} \) leaves the same remainder as \( \frac{n^p}{p} \)

\[ \therefore \text{ } p^{th} \text{ remainder } = \text{ first remainder.} \]

Again \( \frac{n^{p+1}}{p} = \frac{n^{p-1}}{p} \times n^2 = (w + \frac{1}{p}) n^2 = n^2 w + \frac{n^2}{p} \)

\[ \therefore \frac{n^{p+1}}{p} \text{ and } \frac{n^2}{p} \text{ leave the same remainder.} \]

\[ \therefore \frac{a^{p+1}}{p} \text{ and } \frac{a^2}{p} \text{ leave the same remainder.} \]

Generally \( \frac{n^{p+q}}{p} = \frac{n^{p-1}}{p} \times n^{q+1} = (w + \frac{1}{p}) \times n^{q+1} = \)

\[ mw^{q+1} + \frac{n^{q+1}}{p} \]

\[ \therefore \text{ the } (p + q)^{th} \text{ remainder } = \text{ the } (q + 1)^{th} \text{ remainder, } q \]

being any number whatever.

\[ \therefore \text{ after the } (p - 1)^{th} \text{ remainder, they recur.} \]

387. \( a^n = (a - 1 + 1)^n = (a - 1)^n + 1 + n \cdot Q_1 \)

where \( Q \) is integral, since every coefficient of \( (a + b)^n \) expanded

is integral, (see Barlow, p. 177), and \( n \) is prime.

Similarly \( (a - 1)^n = (a - 2)^n + 1 + n \cdot Q_2 \)

&c. = &c.

\[ \left\{ a - (a - 1) \right\}^n = (a - a)^n + 1 + n \cdot Q_3 \]

\[ \therefore a^n = a + n \cdot (Q_1 + Q_2 + Q_3 + \ldots \ldots + Q_n) \]

\[ \therefore \frac{a^n - a}{n} = Q_1 + Q_2 + Q_3 + \ldots \ldots + Q_n = P \]

\[ \therefore a \cdot \frac{(a^{n-1} - 1)}{n} \text{ is divisible by } n; \text{ but } a \text{ is not divisible by } n; \]

\[ \therefore \frac{a^{n-1} - 1}{n} \text{ is an integer (A).} \]

Similarly \( b^{n-1} - 1 \) is an integer (B)

\[ \therefore \frac{a^{n-1} - 1}{n} - \frac{b^{n-1} - 1}{n} = \frac{a^{n-1} - b^{n-1}}{n} \text{ is an integer. Q. E. D.} \]
388. Let \( a = mp + n \) \((n)\) being less than \((a)\).

\[
\begin{align*}
\therefore \quad \frac{a}{p}, \quad \frac{a^2}{p}, \quad \frac{a^3}{p}, & \text{ &c. leave the same remainders as } n, \quad n^2, \quad n^3, \\
& \text{ &c. since every term except the last of the expansions is divisible by } p.
\end{align*}
\]

Now \( \frac{n^p - n}{p} \) is an integer, and since \( n \) is less than, and \( p \) prime to \( p \), \( \frac{n^{p-1} - 1}{p} \) is integral (see preceding problem.)

Put \( \therefore \quad \frac{n^{p-1} - 1}{p} = w \)

\( \therefore \quad \frac{n^{p-1}}{p} = w + \frac{1}{p} \) or we obtain a remainder \( = 1 \) when we take the \( (p-1) \)th term.

389. Let \( \frac{n}{d} \) \( \frac{n'}{d'} \) be the two fractions.

Then \( \frac{n}{d} + \frac{n'}{d'} = \frac{32}{45} = \frac{nd' + n'd}{d'd'} \)

Assume \( dd' = 45 = 5 \times 9 \), and \( \therefore \quad nd' + n'd = 32 \)

But since \( d \) and \( d' \) are prime to each other, \( d = 5 \), and \( d' = 9 \)

\( \therefore \quad 9n + 5n' = 32 \)

\( \therefore \quad n + n' + \frac{4n}{5} = 6 + \frac{2}{5} \)

Put \( \frac{4n}{5} - \frac{2}{5} = w \) (a whole number)

\( \therefore \quad 4n - 2 = 5w \)

\( \therefore \quad n = w + \frac{2}{5} + \frac{2}{5} \)

Put \( \frac{2 + w}{4} = v \) (a whole number)

\( \therefore \quad w = 4v - 2 \)

Hence \( n = 4v - 2 + \frac{2 + 4v - 2}{4} = 5v - 2 \)
PRIME NUMBERS.

and \( n' = \frac{32}{5} - \frac{9n}{5} = \frac{32}{5} - \frac{45v + 18}{5} = 10 - 9v \) in which

values \( v \) may = 0, ± 1, ± 2, ± 3,.....

Let \( v = 0 \)

Then \( n = -2 \) and \( n' = 10 \)

Let \( v = 1 \)

Then \( n = 3 \) and \( n' = 1 \)

&c. &c. &c.
CONTINUED FRACTIONS AND INDETERMINATE PROBLEMS.

\[ \sqrt{2} = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \cdots}}} \]

\[ \sqrt{2} + 1 = 2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \cdots}}} \]

\[ \sqrt{2} + 1 = 2 + \frac{1}{1 + \frac{1}{2 + \frac{1}{2 + \cdots}}} \]

\[ \therefore \sqrt{2} = 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \cdots}}} \]

Now \( \frac{1}{2 + \frac{1}{2}} = \frac{2}{5} \)

and \( \frac{1}{2 + \frac{1}{2}} = \frac{5}{12} \)

\( \frac{1}{\frac{1}{2} + 2} = \frac{12}{29} \)

\( \frac{1}{\frac{1}{2} + 2} = \frac{29}{70} \)

\( \therefore \sqrt{2} = 1 + \frac{2}{5} \text{ nearly, } = 1 + \frac{5}{12} \text{ more nearly} \)

\[ = 1 + \frac{12}{29} \text{ more nearly, &c. &c.} \]
CONTINUED FRACTIONS.

391. \[ \sqrt{3} = 1 + \frac{1}{\sqrt{3} - 1} = 1 + \frac{1}{\frac{\sqrt{3} + 1}{2}} \]
\[ \frac{\sqrt{3} + 1}{2} = 1 + \frac{\sqrt{3} - 1}{2} = 1 + \frac{1}{\frac{\sqrt{3} + 1}{2}} \]
\[ \sqrt{3} + 1 = 2 + \sqrt{3} - 1 = 2 + \frac{1}{\frac{\sqrt{3} + 1}{2}} \]
\[ \frac{\sqrt{3} + 1}{2} = 1 + \frac{\sqrt{3} - 2}{2} = 1 + \frac{1}{\sqrt{3} + 1} \]
\[ \sqrt{3} + 1 = 2 + \sqrt{3} - 1 = 2 + \frac{1}{\sqrt{3} + 1} \]
&c. = &c. = &c.

\[ \therefore \sqrt{3} = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \ldots}}}}}}} \]

Hence the fractions converging to \( \sqrt{3} \) are

\[ \frac{1}{1}, \quad 1 + \frac{1}{1 + \frac{1}{2}}, \quad 1 + \frac{1}{1 + \frac{1}{2 + 1}}, \quad 1 + \frac{1}{1 + \frac{1}{2 + \frac{1}{2 + 1}}}, \quad \&c. \&c. \]

or \( 1, \ 2, \ \frac{5}{3}, \ \frac{7}{4}, \ \frac{16}{9}, \ \&c. \ldots \)

392. \( X^2 - Y^2 = 24 \)
\[ \therefore (X - Y) \times (X + Y) = 1 \times 24 \]
or \( 2 \times 12 \)
or \( 3 \times 8 \)
or \( 4 \times 6 \)
or \( 6 \times 4 \)
or \( 5 \times 3 \)
or \( 12 \times 2 \)
or \( 24 \times 1 \)
CONTINUED FRACTIONS.

\[
\begin{align*}
X + Y &= 24 \quad \{ \quad X + Y &= 12 \quad \{ \quad X + Y &= 8 \quad \{ \quad X + Y &= 6 \quad \{ \\
X - Y &= 1 \quad \{ \quad X - Y &= 2 \quad \{ \quad X - Y &= 3 \quad \{ \quad X - Y &= 4 \quad \{ \\
X + Y &= 4 \quad \{ \quad X + Y &= 8 \quad \{ \quad X + Y &= 2 \quad \{ \quad X + Y &= 1 \quad \{ \\
or \quad X - Y &= 6 \quad \{ \quad X - Y &= 8 \quad \{ \quad X - Y &= 12 \quad \{ \quad X - Y &= 24 \quad \{ \\
\end{align*}
\]

:. the values of \( x \) are \( \frac{25}{2}, \frac{11}{2}, 5, \frac{11}{2}, 7, \) and the corresponding values of \( y \) are \( \frac{-23}{2}, \frac{5}{2}, 1, -1, \frac{-5}{2}, -5. \)

For the complete solution, see Barlow, page 106.

393. \( mx + ny = p \)

\[
\begin{align*}
x &= \frac{p}{m} - \frac{ny}{m} \\
y &= \frac{p}{n} - \frac{mx}{n}
\end{align*}
\]

in which, if \( m \) be prime to \( n \), the least integral values of \( y \) and \( x \), which will render \( x \) and \( y \) integral, are \( m \) and \( n \) respectively; since \( p \) is divisible both by \( m \) and \( n \). If \( m \) and \( n \) be not prime to each other, these are not the least values of \( m \) and \( n \).

Let \( p = qm = nr \), \( \therefore m = \frac{p}{q}, n = \frac{p}{r} \)

\[
\begin{align*}
x &= \frac{p}{q} + \frac{p}{r} y = p \\
y &= r - \frac{x}{q}
\end{align*}
\]

in which the least integral values of \( y \)

and \( x \) are the nearest integers, which, when substituted for \( y \) and \( x \) in \( \frac{q}{r}, y, \frac{r}{q}, x \), render them integral.
394. Let \( x \) be the number required.

Then \( x = 2q + 1 = 3q' + 2 = 5q'' + 3 \) (\( q, q', q'' \) being the quotients corresponding to the divisors, 2, 3, 5, respectively).

Now, these quotients being integral, we must seek integer values of them from the two equations,

\[
2q + 1 = 3q' + 2 \quad (1) \\
5q'' + 3 = 3q' + 2 \quad (2)
\]

\[2q = 3q' + 1 \]
\[5q'' = 3q' - 1 \]

\[\therefore q = q' + \frac{q' + 1}{2} \]

and \( q'' = \frac{3q' - 1}{5} \)

Put \( \frac{q' + 1}{2} = w \) an integer

\[\therefore q' = 2w - 1 \]

\[q'' = \frac{6w - 4}{5} = w + \frac{w - 4}{5} \]

Let \( \frac{w - 4}{5} = v \)

\[\therefore w = 5v + 4, \text{ where } v \text{ may } = 0, 1, 2, \&c.\]

The corresponding values of \( w \) are 4, 9, 14, 19, &c.

and of \( q' \) ..., 7, 17, 27, 37, &c.

and of \( x \) ..., 23, 53, 83, 113, &c.

395. Let \( x \) be the representative of the roots of unity.

Then \( x^3 = 1 = \cos \, 2m\pi + \sqrt{-1} \sin \, 2m\pi \), \( m \) being any integer.
\[ x = \cos \frac{2m\pi}{3} + \sqrt{-1} \sin \frac{2m\pi}{3}, \text{ by De Moivre's Theorem} \]

\[ x^n = \cos \frac{2mn\pi}{3} + \sqrt{-1} \sin \frac{2mn\pi}{3} \quad (n \text{ being integral}) \]

\[ = (\cos 2mn\pi + \sqrt{-1} \sin 2mn\pi)^n = (1)^n = x, \text{ since } 2mn \text{ is an even number.} \]

Again, let \( n \) be fractional and \( = \frac{p}{q} \)

\[ x^{\frac{p}{q}} = \left( \cos 2mp\pi + \sqrt{-1} \sin 2mp\pi \right)^\frac{1}{q} \]

\[ = (\cos 2mp\pi + \sqrt{-1} \sin 2mp\pi)^\frac{1}{q} \quad (\text{since } 2mpq \text{ is even,}) \]

\[ = (\cos 2mp\pi + \sqrt{-1} \sin 2mp\pi)^\frac{1}{q} \quad (\text{by De Moivre.}) \]

\[ = (1)^\frac{1}{q} = x = \text{ cube root of unity.} \]

Generally, it may be shown in the same way, that any power of any root of unity is itself a root of unity.

396. Let \( x \) be one part, then \( 2 - x \) is the other, and
\[ x - 2 + x \text{ or } 2x - 2 = \text{ difference of the parts.} \]

But \( 2 - x + x^2 - (x + (2 - x))^2 = 2 - x + x^2 - (x + 4 - 4x + x^2) = 2 - x + x^2 - 4 + 4x - x^2 = 2x - 2 = \text{ difference of the parts, Q. E. D.} \]

397. Since S, R, Q, P, &c. = sum of products of the roots with their signs changed, taken \( n \) and \( n \), \( n - 1 \) and \( n - 1 \), \( n - 2 \) and \( n - 2 \), &c., together respectively, and the numbers of combinations of \( n \) things taken \( n \) and \( n \), \( n - 1 \) and \( n - 1 \), &c., together are \[1, \frac{n(n-1)}{1.2}, \frac{n(n-1)(n-2)}{1.2.3}, \quad \text{&c. respectively,} \]

\[ \therefore \text{ S consists of one term} \]
\[ \text{ R consists of } n \text{ terms} \]
\[ \text{ Q consists of } \frac{n(n-1)}{1.2} \]
\[ \text{ P consists of } \frac{n(n-1)(n-2)}{1.2.3} \]

&c. &c.

Again, since in \( (n-1) \) roots (all except \( x \)) taken, \( n - 1 \) and
$n-1$, $n-2$ and $n-3$ and $n-3$, &c., there may be formed
\[ \frac{(n-1) \cdot (n-2)}{1 \cdot 2} \]

\[ \therefore \text{R contains only } 1 \text{ term } = (r) \text{ which is not divisible by } (a) \]

\[ \text{Q... contains } (n-1) \text{ terms } = (q) \text{...} \]

\[ \text{P... contains } \frac{(n-1) \cdot (n-2)}{1 \cdot 2} \text{ terms } = (p) \text{...} \]

&c. &c.

Hence S contains one term $= (s \cdot a)$ divisible by $a$,

\[ \text{R contains } n-1 \text{ terms } = (r' \times a) \text{...} \]

\[ \text{Q contains } \frac{n-1}{1 \cdot 2} \text{ terms } = \frac{(n-1) \cdot (n-2)}{1 \cdot 2} \text{ terms} \]

\[ = (q' \times a) \text{...} \]

&c. &c.

Hence $S = sa$

\[ R = r + ar', Q = q + aq', \text{ &c. = &c. &c.} \]

Now $\frac{S}{a} = s = -(\text{product of all the roots except } (a) \text{ with their signs changed}) = -r'$ evidently.

\[ \therefore \frac{S}{a} + R = \frac{S}{a} + r + r' \times a = -r + r + r' \times a = r' a \]

\[ \therefore R' = \frac{r'}{a} \]

Again, $\frac{R'}{a} + Q = r' + q + aq'$

But, since $ar' = -a \times (-r') = -a \times (\text{sum of the products of } (n-1) \text{ roots with their signs changed, taken } (n-2) \text{ together } = -a \times q$

\[ \therefore r' = -q \]

Similarly $q' = -p$

&c. = &c.

\[ \therefore \frac{R'}{a} + Q = aq = Q' \]

\[ \therefore \frac{Q'}{a} = q' \]

\[ \therefore \frac{Q'}{a} + P = \text{an integer } = q' + p + ap' = ap' \]
\[ \therefore \frac{P'}{a} = P' \]

\[ \&c. = \&c. \]

\[ \therefore \frac{S}{a} \] \[ R', Q', P', \&c., \text{ are integers.} \]

N. B. This elegant property of the coefficients which may also be proved by substituting \( a \) for \( x \), \&c. \&c., may be applied to find the roots of equations, provided those roots be integral.

Thus \( x^5 - 8x^4 + 27x^3 - 52x^2 + 56x - 24 = 0 \)

Since \( R' = \frac{S}{a} + R \)

\[ Q' = \frac{R'}{a} + Q = \frac{S}{a^2} + \frac{R}{a} + Q \]

\[ P' = \frac{Q'}{a} + P = \frac{S}{a^3} + \frac{R}{a^2} + \frac{Q}{a} + P \]

\[ \&c. = \&c. \]

\[ \therefore \text{if by trial we find such a value of} \ a \ \text{as shall make} \]

\[ \frac{S}{a^n} + \frac{R}{a^{n-1}} + \frac{Q}{a^{n-2}} + \ldots \to \text{(m terms, or} \frac{S + aR + a^2Q + \ldots}{a^n} \]

an integer, that value will in all probability be a root, and substitution may be made accordingly. Three or four terms will be sufficient for this trial.

In the above equation \( S = -24 \)

\[ R = 56 \]

\[ Q = -24 \]

\[ \therefore \frac{-24 + 3 \times 56 - 9 \times 24}{3^3} = \frac{-324}{27} = -12 \]

\[ \therefore \text{we may conclude} \ 3 \text{to be a root.} \]

Other applications may also be made, which we leave to the ingenuity of the reader.

398. Let \( x \) be the number required.

Then \( x = 3q + 1 = 5q' + 3 \), \( q \) and \( q' \) being the quotients corresponding to the divisors 3 and 5 respectively.

Hence \( 3q = 5q' + 2 \)

\[ \therefore q = 2q' + 1 - \frac{q' + 1}{3} \]
**MISCELLANIES.**

Put \( q' + \frac{1}{3} = w \) an integer.

\[ q' = 3w - 1 \]

and \( q = \frac{15w - 5 + 2}{3} = 5w - 1 \)

Let \( w = 0, 1, 2, &c. \) then \( q = -1, 4, 9, &c. \)

\[ x = -2, 13, 28, &c. \]

\( \therefore \) the least value of \( x \) is 13

399. Let \( a = b + x \). \( \therefore \) \( a - b = x \)

Then \( a^n = b^n + n b^{n-1} x + \ldots \)

\[ \therefore \frac{a^n - b^n}{x} = n b^{n-1} + n \frac{n-1}{2} b^{n-2} x + \ldots \]

\( \therefore a^n - b^n \) is divisible by \( x \) or \( a - b \)

But \( a^n + b^n = 2b^n + n b^{n-1} x + \ldots \) which is not divisible by \( x \), or \( a - b \), unless \( 2b^{n-1} \) is, i.e., unless \( \frac{2b^{n-1}}{a - b} \) or \( \frac{b}{a - b} \) be an integer.

Again, let \( a = x - b \) or \( x = a + b \)

Then \( a^n = x^n - n x^{n-1} b + \ldots \pm b^n \) according as \( n \) is even or odd.

\[ \therefore a^n - b^n = x^n - n x^{n-1} b + \ldots \pm b^n \]

\( \therefore a^n - b^n \) is divisible by \( x \) or \( a + b \), if \( n \) be even, or if \( n \) be odd, provided \( \frac{2b^n}{a + b} \) be an integer, but in all other cases, \( a^n - b^n \) is not divisible by \( a + b \).

400. Let \( 2x + 1 \) represent the odd number, \( x \) being any integer whatever.

Then \((2x + 1)^2 + 3 = 4x^2 + 4x + 4 \) which is evidently divisible by 4.

401. \((2n)^3 = 8n^3 \) which is even.

\((2n + 1)^3 = 8n^3 + 12n^2 + 6n + 1 \) \( = 2(4n^2 + 6n^2 + 3n) + 1 \) which is odd.
MISCELLANIES.

\[ \therefore \text{the root of every odd cube is also odd.} \]

Let \( \therefore 2x + 1 \) be the root of the given cube.

Then the middle term of the series \((2x + 1)^2 = 4x^2 + 4x + 1\)
and the common difference = 1

\[ \therefore \text{the last term} = 4x^2 + 5x + 1 \]
and the first term = \(4x^2 + 3x + 1\)

\[ \therefore \text{the sum of the series} = \left( 8x^3 + 8x + 2 \right) \frac{2x + 1}{2} \]
\[ = (4x^3 + 4x + 1) \cdot (2x + 1) \]
\[ = 8x^3 + 12x^2 + 6x + 1 \]
\[ = (2x + 1)^3 \quad \text{Q.E.D.} \]

402. This is nothing more than finding two squares together = a given square, the sides of those squares being rational.

Let \( a^2 \) be the given square, and \( x^2, y^2 \) those sought.

Then \( a^2 = x^2 + y^2 \)

\[ \therefore a^2 - y^2 = x^2 = (a - y)(a + y) \]

Put \( a + y = \frac{px}{q} \)

and \( a - y = \frac{qx}{p} \)

\[ \therefore 2a = \frac{p^2 + q^2}{pq} x \]

\[ 2y = \frac{p^2 - q^2}{pq} x \]

\[ \therefore x = \frac{2apq}{p^2 + q^2} \times a \]

\[ y = \frac{p^2 - q^2}{p^2 + q^2} \times a \text{ where } p \text{ and } q \text{ may be assumed of any magnitude.} \]

If \( a = \) two squares, put \( p^2 + q^2 = a \) or any factor of it, and we have the above forms integral.

Let \( a = 5 = 2^2 + 1 \)

Then \( p^2 = 2^2, q^2 = 1 \)

\[ \therefore x = \frac{4 \times 5}{5} = 4 \]

\[ y = \frac{1 \times 5}{5} = 1 \]
Let \( a = 30 \)

Then \( x = \frac{2apq}{p^2+q^2} = \frac{60pq}{p^2+q^2} \)

\[ y = \frac{p^2-q^2}{p^2+q^2} \times 30 \quad \text{Let } p = 2 \text{ and } q = 1 \]

\[ \therefore \quad x = \frac{60 \times 2}{5} = 24 \]

\[ y = \frac{3 \times 30}{5} = 18 \]

\[ \therefore \text{if the hypothenuse be 30, the legs will be 24 and 18. Similar results may be found ad libitum.} \]

403. \( \log N = \log (N \times 1) = \log N + \log 1. \)

Let \( e \) be the hyperbolic base.

Then \( \cos \theta + \sqrt{-1} \sin \theta = e^{2\sqrt{-1}} \)

\[ \therefore \cos 2n\pi + \sqrt{-1} \sin 2n\pi = e^{2n\pi \sqrt{-1}} \]

\[ \therefore 1 = e^{2n\pi \sqrt{-1}} \]

\[ \therefore \log (1) = 2n\pi \sqrt{-1} \text{ where } n \text{ may be any number from 0 to } \infty; \text{ when } n = 0, \log 1 = 0. \]

\[ \therefore \log N = \log N \text{ and is real in this case only, since all the other values involve } \sqrt{-1}. \]

Again, \( \log (-N) = \log N + \log (-1) \)

But since \( \cos \theta + \sqrt{-1} \sin \theta = e^{\sqrt{-1}} \)

and \( \therefore \cos (2n+1)\pi \sqrt{-1} \sin (2n+1)\pi \sqrt{-1} = \)

\[ \therefore -1 = e^{(2n+1)\pi \sqrt{-1}} \]

\[ \therefore \log (-1) = (2n+1)\pi \sqrt{-1} \text{ which for every value of } n \text{ is imaginary. } \]

Hence \( \log (-N) = \log N + \log (-1) \) is always imaginary.
ARITHMETIC OF SINES.

404. Let \( (a) \) be the given angle, \( (x) \) one of the parts and \( (a-x) \) the other part.

Then \( \tan \frac{(a-x)}{\tan x} = \frac{n}{1} \)

\( \therefore \tan a - \tan x \)

\( 1 + \tan a \cdot \tan x = n \tan x \)

\( \therefore \tan a - \tan x = n \tan x + n \tan a \cdot \tan x \)

\( \therefore \tan x + \frac{n+1}{n \tan a} \tan x = \frac{1}{n} \)

\( \therefore \tan x = \pm \sqrt{(n+1)^2 + \frac{1}{n} - \frac{n+1}{2n \tan a}} \)

But \( \frac{(n+1)^2}{4n^2 \tan^2 a} + \frac{1}{n} = \frac{n^2 + 2n + 1 + 4n \tan^2 a}{4n^2 \tan^2 a} \)

\( = \frac{n^2 - 2n + 1 + 4n(1 + \tan^2 a)}{4n^2 \tan^2 a} \)

\( = \frac{(n-1)^2 + 4n \cdot \sec^2 a}{4n^2 \tan^2 a} \)

\( \therefore \tan x = \pm \sqrt{(n-1)^2 + 4n \cdot \sec^2 a - (n+1)} \)

\( \therefore \tan x \) is known, and, by reference to the table we shall obtain \( x \). Hence \( a-x \) is known.

The \( \tan x \) may easily be constructed, and thence will appear the geometrical solution of the problem.

405. \( \tan \frac{A}{2} = \frac{\sin \frac{A}{2}}{\cos \frac{A}{2}} = \frac{2 \sin \frac{A}{2} \cos \frac{A}{2}}{2 \cos^2 \frac{A}{2}} \)

\( = \sin A \).
ARITHMETIC OF SINES.

Since \( \cos A = \cos^2 \frac{A}{2} - \sin^2 \frac{A}{2} = 2 \cos^2 \frac{A}{2} - 1 \), &c.

This may be proved geometrically by taking in the circle whose radius \( AB \), or \( AC \) = unity, the \( \angle BAC = \angle A \), &c.

Thus, produce \( CA \) to \( D \), join \( DB \), and bisect \( BAC \) by the radius \( AE \), and draw the tan. \( ET \) meeting \( AB \) produced in \( T \). Also draw \( TN \) perpendicular \( AC \).

Then \( \angle TAE = \frac{1}{2} \angle TAC = \angle D \)

\[ \therefore \text{AE is parallel to DT.} \]

and the triangles \( TAE \), \( TDN \) are \( \therefore \) similar.

\[ \therefore \text{TE} : \text{AE} :: \text{TN} : \text{DN} \]

or \( \tan \frac{A}{2} : 1 :: \sin A : 1 + \cos A. \therefore \text{&c.} \)

406. \( \sin (a + b) = \sin a \cdot \cos b + \cos a \cdot \sin b \) and \( \sin (a - b) = \sin a \cdot \cos b - \cos a \cdot \sin b \) \( \frac{1}{2} \) (See Woodhouse or Creswell), and generally \( (x + y) \cdot (x - y) = x^2 - y^2 \).

\[ \therefore \sin (a + b) \cdot \sin (a - b) = \sin^2 a \cdot \cos^2 b - \cos^2 a \cdot \sin^2 b = \sin^2 a \cdot (1 - \sin^2 b) - (1 - \sin^2 a) \cdot \sin^2 b = \sin^2 a - \sin^2 b. \]

The geometrical proof will not be difficult.

407. \( \cos 2a = \cos (a + a) = \cos a \cdot \cos a - \sin a \cdot \sin a = \cos^2 a - \sin^2 a = \cos^2 a - (1 - \cos^2 a) = 2 \cos^2 a - 1. \)

408. \( \frac{\sin (a - b)}{\sin a \cdot \sin b} + \frac{\sin (b - c)}{\sin b \cdot \sin c} + \frac{\sin (a - c)}{\sin a \cdot \sin c} = \frac{\sin c \cdot \sin (a - b) + \sin a \cdot \sin (b - c) + \sin b \cdot \sin (a - c)}{\sin a \cdot \sin b \cdot \sin c} \)

But \( \sin c \cdot \sin (a - b) + \sin a \cdot \sin (b - c) + \sin b \cdot \sin (a - c) = 0. \)

For \( \sin c \cdot \sin a \cdot \cos b - \cos a \cdot \sin b \cdot \sin c \)

\[ + \sin a \cdot \sin b \cdot \cos c - \cos b \cdot \sin c \cdot \sin a \]

\[ + \sin b \cdot \sin a \cdot \cos c - \cos c \cdot \sin a \cdot \sin b \] = 0.
ARITHMETIC OF SINES.

\[
\therefore \frac{\sin (a-b)}{\sin a} + \frac{\sin (b-c)}{\sin b} + \frac{\sin (a-c)}{\sin c} = 0, \text{ if the common denominator } \sin a, \sin b, \sin c \text{ be finite; i.e. if } a, b, c \text{ be each } > 0, \text{ and } < \pi, > \pi \text{ and } < 2 \pi, \text{ &c. &c.}
\]

409. Let \( x \) be the arc required.

Then \( \cos x = \tan x = \frac{\sin x}{\cos x} \) to radius = 1.

\[
\therefore \cos^2 x = \sin x
\]

\[
\therefore 1 - \sin^2 x = \sin x
\]

\[
\therefore \sin^2 x + \sin x = 1
\]

and \( \sin^2 x + \sin x + \frac{1}{4} = 1 + \frac{1}{4} = \frac{5}{4} \)

\[
\therefore \sin x = \frac{-1 \pm \sqrt{5}}{2}
\]

\[
\therefore x \text{ is that arc whose sine } = \frac{-1 + \sqrt{5}}{2}, \text{ or that whose sine } = \frac{-1 - \sqrt{5}}{2}, \text{ the radius being unity. Reduce these surd values to decimals, and refer to the tables (Hutton's), where are to be found the rules necessary for finding the arcs corresponding to the two sines.}
\]

410. Let \( a \) be the given \( \angle \), \( x \) one of the parts required, and \( \therefore a - x \) the other part.

Then \( \frac{\sin (a-x)}{\sin x} = \frac{n}{1} \frac{n}{1} \) being the given ratio.

\[
\therefore \sin a \cdot \cos x - \cos a \cdot \sin x = n \sin x
\]

\[
\therefore \sin a \cdot \frac{\cos x}{\sin x} = \cos a + n
\]

\[
\therefore \cot x = \frac{\cos a}{\sin a} + n = \cot a + n \cosec a
\]

whence \( x \) is known, by reference to the tables.

\[
\therefore a - x \text{ is also known.}
\]
ARITHMETIC OF SINES.

410. Let the two arcs be $a$ and $b$

Then

$$\frac{\cos a + \cos b}{\cos a - \cos b} = \frac{2 \cos \frac{a+b}{2} \cos \frac{a-b}{2}}{2 \sin \frac{a+b}{2} \sin \frac{a-b}{2}} = \frac{\cot \frac{a+b}{2}}{\tan \frac{a-b}{2}}$$

For $\cos (A + B) + \cos (A - B) = 2 \cos A \cos B = 2 \times$

$$\frac{A + B + (A - B)}{2} \times \sin \frac{A + B - (A - B)}{2}$$

and $\cos (A + B) - \cos (A - B) = 2 \sin A \sin B = \&c.$

\[\therefore \cos a + \cos b : \cos a - \cos b :: \cot \frac{a+b}{2} : \tan \frac{a-b}{2}\]

411. $\sin (60+A) = \sin 60 \cdot \cos A + \cos 60 \cdot \sin A$.

$\sin (60-A) = \sin 60 \cdot \cos A - \cos 60 \cdot \sin A$.

\[\therefore \sin (60+A) - \sin (60-A) = 2 \sin A \cdot \cos 60 = \sin A\]

\[\therefore \sin (60+A) = \sin (60 - A) + \sin A.\]

412. $\frac{\sin a + \sin b}{\cos a + \cos b} = \frac{2 \sin \frac{a+b}{2} \cos \frac{a-b}{2}}{2 \cos \frac{a+b}{2} \cos \frac{a-b}{2}} = \frac{\sin \frac{a+b}{2}}{\cos \frac{a+b}{2}}$

\[= \tan \frac{a+b}{2}\]

For $\sin (A + B) + \sin (A - B) = 2 \sin A \cdot \cos B$

and $\cos (A + B) + \cos (A - B) = 2 \cos A \cdot \cos B$.

i.e. the sum of the sines of two arcs $= 2 \times$ (the sin. of half their sum) $\times$ (cos. of half their difference), and the sum of the cosines of two arcs $= 2 \times$ (cos. of half their sum) $\times$ (cos. of half their difference).

413. $\tan 2A = \tan (A + A) = \frac{\tan A + \tan A}{1 - \tan A \tan A} = \frac{2 \tan A}{1 - \tan^2 A}$

Let $\tan 2A = 2 \tan A + x$

\[\therefore x + 2 \tan A = \frac{2 \tan A}{1 - \tan^2 A}\]
\[
\therefore x = \frac{2 \tan A - 2 \tan A + 2 \tan^3 A}{1 - \tan^3 A}
\]

which is positive when \(\tan A\) is positive, and less than unity (the radius), or negative and greater than unity; i.e. when \(A\) is positive and less than \(45^\circ\), or when it is negative and greater than \(45^\circ\).

\[\therefore 2A\text{ is greater than } 2 \tan A\text{ when } A\text{ is positive and less than } 45^\circ, \text{ or when } A\text{ is negative and greater than } 45^\circ.\]

\(\text{N. B. If an arc lying on one side of the diameter be called positive, the arc adjacent on the other side of the diameter is termed negative.}\)

The limits of \(2A\) with respect to \(2 \tan A\), may similarly be found, for the other quadrants. Also when \(\tan 2A\) is less than \(2 \tan A\). This we leave to the reader.

414. Let \(2 \cos \theta = u + \frac{1}{u}\)

Then, if \(2 \cos (m - 1) \theta = u^{m-1} + \frac{1}{u^{m-1}}\)

and \(2 \cos m \theta = u^m + \frac{1}{u^m}\) we shall have

\[2 \cos (m + 1) \theta = u^{m+1} + \frac{1}{u^{m+1}}\]

For \(\cos (A + B) + \cos (A - B) = 2 \cos A \cos B\).

Put \(A = m \theta, B = \theta\)

Then \(\cos (m + 1) \theta + \cos (m - 1) \theta = 2 \cos m \theta \cos \theta\)

\[\therefore \cos (m + 1) \theta = (u^m + \frac{1}{u^m}) (u + \frac{1}{u}) \times \frac{1}{2} - (u^{m-1} + \frac{1}{u^{m-1}}) \times \frac{1}{2}\]

\[\therefore 2 \cos (m + 1) \theta = u^{m+1} + \frac{1}{u^{m+1}} + u^{m-1} + \frac{1}{u^{m-1}} - u^{m-1} - \frac{1}{u^{m-1}}\]
if the proposition be true for any two successive values of \( m \) (\( m \) being integral), it is also true for the next higher value of \( m \).

But
\[
2 \cos^2 \theta = 2 \left( \cos^2 \theta - 1 \right) = 4 \cos^2 \theta - 2
\]

\[
= u^2 + 2 + \frac{1}{u^2} - 2 = u^2 + \frac{1}{u^2}
\]

and \( 2 \cos \theta = u + \frac{1}{u} \) by supposition.

\[
2 \cos^3 \theta = u^3 + \frac{1}{u^3}
\]

\&c.

\[
2 \cos n \theta = u^n + \frac{1}{u^n}\text{ if } n \text{ be integral. For, greater information on the subject, see Woodhouse's Trigonometry.}
\]

\[
\text{Otherwise,}
\]

Since \( u^2 - 2 u \cos \theta = -1 \), solve the equation in \( u \).

\[
\therefore \quad u = \cos \theta + \sqrt{\cos^2 \theta - 1} = \cos \theta + \sqrt{-1} \sin \theta
\]

Hence
\[
\frac{1}{u} = \frac{1}{\cos \theta + \sqrt{-1} \sin \theta} = \frac{\cos \theta - \sqrt{-1} \sin \theta}{\cos^2 \theta + \sin^2 \theta}
\]

\[
= \cos \theta - \sqrt{-1} \sin \theta
\]

Hence we have
\[
u^n = (\cos \theta - \sqrt{-1} \sin \theta)^n = \cos n \theta + \sqrt{-1} \sin n \theta
\]

and \( \frac{1}{u^n} = (\cos \theta - \sqrt{-1} \sin \theta)^n = \cos n \theta - \sqrt{-1} \sin n \theta\)

\[
\therefore \quad u^n + \frac{1}{u^n} = 2 \cos n \theta
\]

and \( u^n - \frac{1}{u^n} = 2 \sqrt{-1} \sin n \theta \), where \( n \) may have any value whatever.

415. \[
\tan (45 + A) = \frac{\tan 45 + \tan A}{1 - \tan 45 \times \tan A} = \frac{1 + \tan A}{1 - \tan A}
\]
\[
\tan (45 - A) = \frac{\tan 45 - \tan A}{1 + \tan 45 \times \tan A} = \frac{1 - \tan A}{1 + \tan A}
\]
\[
\therefore \tan (45 + A) - \tan (45 - A) = \frac{1 + \tan A}{1 - \tan A} - \frac{1 - \tan A}{1 + \tan A} = \frac{1 + 2\tan A + \tan^2 A}{1 - \tan^2 A} - \frac{1 + 2\tan A - \tan^2 A}{1 - \tan^2 A}
\]

\[
= \frac{4\tan A}{1 - \tan^2 A}
\]

But \(2\tan 2A = 2\tan (A + A) = 2 \times \frac{\tan A + \tan A}{1 - \tan A \tan A} = \frac{4\tan A}{1 - \tan^2 A}
\]

\[\therefore \tan (45 + A) - \tan (45 - A) = 2\tan 2A\]

416. Let the arc be \(A\).

Then \(\tan A + \cot A = \frac{\sin A}{\cos A} + \frac{\cos A}{\sin A} = \frac{\sin^2 A + \cos^2 A}{\sin A \cos A} = \frac{1}{\sin A \cos A}
\]

\[
= \frac{1}{\sin 2A} = \frac{2}{\sin 2A} = 2 \csc 2A
\]

since \(\sin 2A = 2 \sin A \cos A\), and \(\csc 2A = \frac{1}{\sin 2A}\)

417. Let \(A\) be the angle.

Then \(\tan A - \cot A = \frac{1}{\tan A} - \frac{1}{\cot A} = \frac{\tan^2 A - 1}{\tan A} = \pm \frac{1 - \tan^2 A}{\tan A}, \) according as \(\tan A\) is less or greater than unity (the radius).

But \(\cot 2A = \frac{1}{\tan 2A} = \frac{1}{2 \tan A} = \frac{1 - \tan^2 A}{2 \tan A}
\]

\[\therefore \tan A - \cot A = \pm 2 \cot 2A.
\]

418. By the above problem, we have

\(\tan A = \cot A \pm 2 \cot 2A\), according as \(\tan A\) is less or greater than the radius, i.e. as \(A\) is \(<\) or \(>\) 45°
ARITHMETIC OF SINES.

\[ \therefore \tan 50^\circ = \cot 50^\circ - 2 \cot (100^\circ) \]

But \( \cot 100^\circ = \cot (90 + 10) = \frac{1 - \tan 90 \times \tan 10}{\tan 90 + \tan 10} \)

\[ = \frac{\infty \times \tan 10}{\infty} = - \tan 10 \]

and \( \cot 50 = \tan 40 \)

\[ \therefore \tan 50 = \tan 40 + 2 \tan 10. \]

419. Let \( \tan A \) and \( \tan B \), be the given tangents.

Then \( \tan (A \pm B) = \frac{\sin (A \pm B)}{\cos (A \pm B)} = \frac{\sin A \cos B \pm \sin B \cos A}{\cos A \cos B \mp \sin A \sin B} \)

Divide by \( \cos A \cos B \)

Then \( \tan (A \pm B) = \frac{\sin A \pm \sin B}{\cos A \cos B} \cdot \frac{1 \mp \tan A \tan B}{\sin A \sin B} \)

420. Let the radius = unity.

Then \( \cos (A + B) = \cos A \cos B - \sin A \sin B \)

and \( \cos (A - B) = \cos A \cos B + \sin A \sin B \)

\[ \therefore \cos (A + B) + \cos (A - B) = 2 \cos A \cos B \]

and to introduce radius \( r \), we must divide each function of the arcs by \( r \). (see Woodhouse.)

\[ \frac{\cos (A + B)}{r} + \frac{\cos (A - B)}{r} = \frac{2 \cos A}{r} \times \frac{\cos B}{r} \]

\[ \frac{\cos (A + B)}{2} + \frac{\cos (A - B)}{2} = \frac{\cos A \times \cos B}{r} \]

421. Let \( A \) be the arc whose sine is \( s \)

Then \( \sin A = s \)

\[ \cos A = \sqrt{1 - s^2} \]

\[ \text{vers. } A = 1 - \cos A = 1 - \sqrt{1 - s^2} \]
\[ \sec A = \frac{1}{\cos A} = \frac{1}{\sqrt{1 - s^2}} \]
\[ \tan A = \frac{\sin A}{\cos A} = \frac{s}{\sqrt{1 - s^2}} \]
\[ \cot A = \frac{\cos A}{\sin A} = \frac{\sqrt{1 - s^2}}{s} \]

422. \quad \tan (A + B + C) = \tan m \pi = 0

But \( \tan (A + B + C) = \frac{\tan A + \tan (B + C)}{1 - \tan A \times \tan (B + C)} \)

\[ \therefore \tan A + \tan (B + C) = 0 \]

or \[ \tan A + \frac{\tan B + \tan C}{1 - \tan B \times \tan C} = 0 \]

\[ \therefore \tan A + \tan B + \tan C = \tan A \times \tan B \times \tan C = 0 \]

423. \quad \text{Let } A, B, C \text{ be the three parts of the quadrant.}

Then \( \tan A \times \tan B + \tan A \times \tan C + \tan B \times \tan C = r^2 \)

For \( \tan A \times \tan C + \tan B \times \tan C = (\tan A + \tan B) \times \cot (A + B) \), since \( A + B + C = 90^\circ \)

and \( \cot (A + B) = \frac{1 - \tan A \times \tan B}{\tan A + \tan B} \)

\[ \therefore \tan A \times \tan B + \tan A \times \tan C + \tan B \times \tan C \]

\[ \begin{align*}
&= \tan A \times \tan B + 1 - \tan A \times \tan B = 1, \text{ to radius unity.} \\
&\text{or } \tan A \times \tan B + \tan A \times \tan C + \tan B \times \tan C = 1 \\
&\text{to radius } r \\
&\therefore \tan A \times \tan B + \tan A \times \tan C + \tan B \times \tan C = r^2 
\end{align*} \]

424. \quad \text{Let } A + B + C = (2n + 1) \frac{\pi}{2}, \text{ where } A, B, C \text{ are}

the parts of the odd multiple \((2n+1)\) of \(\frac{\pi}{2}\).
ARITHMETIC OF SINES.

Then \( \cot \left( \frac{2n+1}{2} \right) \pi = \cot \left( n\pi + \frac{\pi}{2} \right) \)

\[
= 1 - \tan n\pi \times \tan \frac{\pi}{2}
\]

\[
= \tan n\pi + \tan \frac{\pi}{2}
\]

\[
= \frac{1 - 0 \times \infty}{0 + \infty} = \frac{1 - \text{finite quantity}}{\infty} = 0
\]

\[
\therefore \cot (A + B + C) = \frac{\cot (A+B) \times \cot C - 1}{\cot (A+B) + \cot C} = 0
\]

\[
\therefore \cot (A+B) \times \cot C = 1
\]

But \( \cot (A+B) = \frac{\cot A \times \cot B - 1}{\cot A + \cot B} \)

\[
\therefore \cot A \times \cot B - 1 \times \cot C = 1
\]

or \( \cot A \times \cot B \times \cot C - \cot C = \cot A + \cot B \)

\[
\therefore \cot A + \cot B + \cot C = \cot A \times \cot B \times \cot C
\]

425. \( \cos a + \sqrt{-1} \sin a \times (\cos b + \sqrt{-1} \sin b) \)

\[
= \cos a \times \cos b + \sqrt{-1} \times (\sin a \times \cos b + \cos a \times \sin b) - \sin a \times \sin b = \cos (a+b) + \sqrt{-1} \sin (a+b)
\]

Similarly \( \cos (a+b) + \sqrt{-1} \sin (a+b) \times (\cos c + \sqrt{-1} \sin c) = \cos (a+b+c) + \sqrt{-1} \sin (a+b+c) \), &c. &c.

\[
\therefore \cos (a+b+c+\ldots) + \sqrt{-1} \sin \left( a+b+c+\ldots \right) = \cos a + \sqrt{-1} \sin a \times \cos b + \sqrt{-1} \sin b \times \cos c + \sqrt{-1} \sin c \), &c.
\]

or, dividing and multiplying by \( \cos a, \cos b, \&c. \)

\[
\cos (a+b+c+\ldots) \times \sqrt{-1} \tan a \times \sqrt{-1} \tan b \times \sqrt{-1} \tan c + \ldots \text{by the theory of equations.}
\]

But \( \sqrt{-1} \times A = \sqrt{-1} A \)

\[
\sqrt{-1} \times \sqrt{-1} B = - B
\]
\[
\sqrt{-1} \times \sqrt{-1} \times \sqrt{-1} C = -\sqrt{-1} C \\
\sqrt{-1} \times \sqrt{-1} \times \sqrt{-1} \times \sqrt{-1} D = D \\
\]
\[
\cos \frac{(a+b+c+\ldots)}{\cos a \cos b \cos c} + \sqrt{-1} \tan (a + b + c + \ldots) \times \\
\cos (a+b+c+\ldots) = 1 - B + D - F + \ldots + \sqrt{-1} \times (A-C+E-G+\ldots) \\
C + E - \ldots \]

.: equating the real and imaginary quantities, we have
\[
\cos \frac{(a+b+c+\ldots)}{\cos a \cos b \cos c} = 1 - B + D - F + \ldots \\
\]
and \[
\tan (a+b+c+\ldots) \times (1-B+D-F+) = A-C+E-G+\ldots \\
\]

.: \tanh (a+b+c+\ldots) = \frac{A-C+E-G+\ldots}{1-B+D-F+\ldots} \]

Hence, also we have \[
\cos (a+b+c+\ldots) = \cos a \cos b \cos c \ldots \{1-B+D-F+\ldots\} \]
and \[
\sin (a+b+c+\ldots) = \tan \times \cos \ldots = \cos a \cos b \cos c \ldots \{A-C+E-G+\ldots\} \]

Many curious, and perhaps, useful propositions may hence be established.

Required to express \(\tan (a+b+c+\ldots)\) in terms of \(\tan (2a), \tan (2b), \text{ &c.}\), or, generally, in terms of \(\tan (ma) \tan (mb), \tan (mc), \text{ &c.}\).

Let \(S_1\) = sum of tangents of \(a, b, c, \text{ &c.}\).
\[
S_1 = \text{sum of tangents of } 2a, 2b, 2c, \text{ &c.} \\
\]
\[
\text{ &c.} = \text{ &c.} \\
S_2 = \text{sum of products of every two of tangents of } a, b, c, \text{ &c. &c.}, \text{ and generally, let } S_n \text{ express the sum of the products of every } n \text{ of the quantities } \tan (ma), \tan (mb), \tan (mc), \text{ &c.} \]

Then by the above form we have
\[
\tan (m\phi) = \frac{m \tan \phi - m \cdot \frac{m-1}{2} \cdot \frac{m-2}{3}, \tan^2 \phi + \text{ &c.}}{1 - m \cdot \frac{m-1}{2} \tan^2 \phi + m \cdot \frac{m-1}{2} \cdot \frac{m-2}{3} \cdot \frac{m-3}{4}, \tan^4 \phi + \text{ &c.}} \]
Also \( \tan{(ma + mb + mc + \ldots)} = \frac{S_1 - S_3 + S_5 - \ldots}{1 - S_2 + S_4 - \ldots} \)

Put \( a + b + c + \ldots = \phi \)

Then \( \tan{m\phi} = \frac{S_1 - S_3 + S_5 - \ldots}{1 - S_2 + S_4 - \ldots} \)

\[
m \tan{\phi} - m \cdot \frac{m-1}{2} \cdot \frac{m-2}{3} \tan^2{\phi} + \&c.
\]

\[
= \frac{1 - m \cdot \frac{m-1}{2} \tan^2{\phi} + \&c.}{2}
\]

\[
:\therefore \text{we have an equation involving} \tan{\phi} \text{ or} \tan{(a+b+c+\ldots)} \text{ and} \tan{(ma)}, \tan{(mb)}, \tan{(mc)}, \&c., \text{and consequently by solving that equation, or finding the values of} \tan{\phi}, \text{we can express} \tan{\phi} \text{ in terms of} \tan{(ma)}, \tan{(mb)}, \&c.
\]

Let \( m = 2 \)

Then \( \tan{2\phi} = \frac{2}{1 - \tan^2{\phi}} \)

\[
\text{Let} \quad \frac{S_1 - S_3 + S_5 - \ldots}{1 - S_2 + S_4 - \ldots} = Q
\]

\[
:\therefore Q - Q \times \tan^2{\phi} = 2 \tan{\phi}
\]

\[
:\therefore \tan^2{\phi} + \frac{2}{Q} \tan{\phi} = 1
\]

\[
:\therefore \tan{\phi} = \frac{-1 \pm \sqrt{Q^2 + 1}}{Q}
\]

If \( m = 3 \), a cubic equation must be solved, \&c. \&c.

426. This is Demoivre's Theorem, which is deducible from one more general, viz. \( \cos{(A \pm B \pm C \pm \ldots)} \pm \sqrt{-1} \times \sin{(A \pm B \pm C \pm \ldots)} = (\cos{A} \pm \sqrt{-1} \sin{A}) \times (\cos{B} \pm \sqrt{-1} \sin{B}) \times (\cos{C} \pm \sqrt{-1} \sin{C}) \times \&c. \) to prove which we proceed as follows:

Put \( \cos{A} = a, \cos{A} = a' \)

\[
\cos{B} + \sqrt{-1} \sin{B} = a, \cos{B} = a' \cos{B} \sin{A} + \cos{A} \sin{B} + \sqrt{-1} (\sin{A} \times \cos{B} + \cos{A} \sin{B})
\]

\[
\&c. = \&c.
\]
\[ = \cos \left( A + B \right) + \sqrt{-1} \sin \left( A + B \right) \]

Similarly \( a \times b \times c = \cos \left( A + B + C \right) + \sqrt{-1} \sin \left( A + B + C \right) \)

\[ \&c. = \&c. \]

(1) and \( a \times b \times c \times d \times \&c. = \cos \left( A + B + C + D + \ldots \right) + \sqrt{-1} \sin \left( A + B + C + D + \ldots \right) \)

Again \( a' \times b' = \cos A \cos B - \sin A \sin B - \sqrt{-1} \) \( (\sin A \cos B + \cos A \sin B) \)

\[ = \cos \left( A + B \right) - \sqrt{-1} \sin \left( A + B \right) \]

Similarly \( a' \times b' \times c' = \cos \left( A + B + C \right) - \sqrt{-1} \sin \left( A + B + C \right) \)

\[ \&c. = \&c. \]

(2) and \( a' \times b' \times c' \times d' \times \&c. = \cos \left( A + B + C + \&c. \right) - \sqrt{-1} \sin \left( A + B + C + \&c. \right) \)

Again \( a \times b' = \cos A \cos B + \sin A \sin B + \sqrt{-1} \) \( (\sin A \cos B - \cos A \sin B) \)

\[ = \cos \left( A - B \right) + \sqrt{-1} \sin \left( A - B \right) \]

or \[ = \cos \left( B - A \right) - \sqrt{-1} \sin \left( B - A \right) \]

From (1) it appears that when all the factors are positive, the result is of the same form with any one of the factors; the angle in both of its terms being the sum of the angles in the factors, and both terms positive.

From (2) we learn, that when each of the factors is negative in the second term, the result is also negative in the second term, and the angle in each of its terms is the sum of the angles in the factors.

But from (3) we find that when one factor is positive in the second term, and the other negative, the result of these two factors is positive or negative in the second term, according as the first factor (arranging the angles in the order of the factors) is positive or negative in its second term; and the \( \angle \) in this result = the difference of the angles of the factors.

Hence then it is manifest that, generally, \( \cos \left( A \pm B \pm C \pm \ldots \right) \)

\[ \pm \sqrt{-1} \sin \left( A \pm B \pm C \pm \ldots \right) = \left( \cos A \pm \sqrt{-1} \sin A \right) \]
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\[ \times (\cos B \pm \sqrt{-1} \sin B) \times (\cos C \pm \sqrt{-1} \sin C) \times \text{&c.,} \]

where the signs are taken according to the above-mentioned circumstances.

The above very general theorem is more so than it appears to be, inasmuch as it comprehends the form \( \cos (A \pm B \pm \&c. \pm A' \pm B' \pm \&c.) \pm \sqrt{-1} \sin (A \pm B \pm \&c. \pm A' \pm B' \&c.) \)

\[ = \frac{1}{\cos A' \pm \sqrt{-1} \sin A'} = \frac{\cos A'}{\cos^2 A' + \sin^2 A'} = \cos A' \pm \sqrt{-1} \sin A' \]

Now, let \( A = \pm B = \pm C = \pm D &c. \) to \( m \) terms

Then \( \cos (m A) \pm \sqrt{-1} \sin (m A) = (\cos A \pm \sqrt{-1} \sin A)^m \times \&c. \) to \( m \) terms = \( (\cos A \pm \sqrt{-1} \sin A)^m \), the signs being taken + and +, - and - only, which is the solution required when \( m \) is integral.

The case when \( m \) is any rational fractional of the form \( \frac{p}{q} \), may be proved as follows:

Let \( A = \frac{p}{q} \times \theta \)

then \( pA = q \theta \)

and \( (\cos A \pm \sqrt{-1} \sin A)^x = \cos pA \pm \sqrt{-1} \sin pA = \cos q \theta = (\cos \theta \pm \sqrt{-1} \sin \theta)^x \)

\[ = (\cos A \pm \sqrt{-1} \sin A)^{\frac{p}{q}} = \cos \frac{pA}{q} \pm \sqrt{-1} \sin \frac{pA}{q} \]

Let now \( m \) be irreducible, and of the form \( \frac{m}{n} \). Then, by a similar train of reasoning we shall prove the truth of this case, and also that of any other which may present itself. This we leave to the student.

Otherwise.

By expanding \( \sin \theta \) and \( \cos \theta \) and \( e^{\pm \sqrt{-1} \theta} \) it will be seen that \( \cos \theta \pm \sqrt{-1} \sin \theta = e^{\pm \theta \sqrt{-1}} \) (\( e \) being the hyperbolic base.)
\[
\therefore \cos (A \pm B \pm C \pm \ldots) \pm \sqrt{-1} \sin (A \pm B \pm C \pm \ldots) \\
= e^{\pm (A \pm B \pm C \pm \ldots) \sqrt{-1}} \times e^{\pm B \sqrt{-1}} \times \&c. \\
= (\cos A \pm \sqrt{-1} \sin A) \times (\cos B \pm \sqrt{-1} \sin B) \times \&c.
\]
as before.

This method proves the problem in one step, thus,

\[
\cos m A \pm \sqrt{-1} \sin m A = e^{\pm m A \sqrt{-1}} = (e^{\pm A \sqrt{-1}})^m = (\cos A \pm \sqrt{-1} \sin A)^m
\]

whatever may be the value of \(m\).

It is not, however, founded on such obvious principles as the preceding method.

427. The sine of an arc is the perpendicular let fall from one extremity of the arc, upon the diameter passing through the other.

By the perpendiculars let fall from each extremity of the arc upon the diameters passing through the other extremity, two right-angled triangles will be formed, having one angle at the centre common to the triangles, and their hypotenuses being radii of the circle, will be equal \(\therefore\) the sides opposite equal \(\angle\) are equal; or the sines are equal.

428. Let the \(\sin A = a\) be given.

Then \(\cos 2A = \cos (A + A) = \cos A \cos A - \sin A \sin A;\)

\[= \cos^2 A - \sin^2 A = 1 - 2 \sin^2 A \]

\[= 1 - 2a^2, \text{ the radius being supposed equal to unity,}\]

429. The chord of an arc = \(2 \sin\) of half that arc.

\[\therefore \text{chord of } 120 = 2 \sin 60 = \frac{\sin 60}{(\frac{1}{2})} = \frac{\sin 60}{\cos 60} = \tan 60\]
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430. The cosines will be sides opposite equal angles in similar right-angled triangles, formed by the radii \( r, R \), cosines \( c, C \), and sines \( s, S \).

But the versed sines \( v, V \), are equal to the differences between the radii and cosines.

\[
\begin{align*}
\text{and } r : R &:: c : C \text{ from similar triangles} \\
\therefore r - c : R - C &:: r : R \\
or v : V &:: r : R.
\end{align*}
\]

431. Since the sine lies on the same side of the diameter as the arc, with respect to that diameter, it must have the same sign as the arc.

\[
\therefore \sin (-A) = -\sin A.
\]

The cosine is identically the same for \((-A)\) as for \(+A\).

\[
\therefore \cos (-A) = \cos A.
\]

The secant is referred to the centre and not to the diameter, and therefore suffers no change of sign with regard to that of the arc.

\[
\therefore \sec (-A) = \sec A.
\]

Otherwise,

\[
\sin (-A) = \sin (0-A) = \sin 0 \times \cos A - \cos 0 \times \sin A = -1 \times \sin A = -\sin A
\]

\&c. \&c. \&c.

Otherwise,

A variable quantity cannot change sign without passing through zero, or infinity, and vice versa.

Hence, the sine passing through zero, when \( A \) changes its sign, changes its sign also.

The cosine does not pass through zero during such change, and \( \therefore \) does not change its sign.

The secant never = zero; it = \( \infty \) when \( A = 90 \), and afterwards changes sign, \&c. \&c.
432. The \( \cot (A \pm B) = \frac{\cos (A \pm B)}{\sin (A \pm B)} \)

\[ = \frac{\cos A \cdot \cos B \mp \sin A \cdot \sin B}{\sin A \cdot \cos B \pm \cos A \cdot \sin B} \]

Divide both denominator and numerator by \( \sin A \cdot \sin B \).

and we get \( \cot (A \pm B) = \frac{\cot A \cdot \cot B \mp 1}{\cot B \pm \cot A} \)

\[ \therefore \cot (A \pm B) = \frac{\cot A}{r} \cdot \frac{\cot B}{r} \mp \frac{1}{r} = \frac{\cot A \cdot \cot B \mp r^2}{\cot B \pm \cot A} \]

\[ \therefore \cot (A \pm B) = \frac{\cot A}{r} \cdot \frac{\cot B}{r} \mp r^2 \]

The rule for introducing the radius \( r \) is obvious.

It is, multiply every term by that power of \( r \) whose index = difference between the highest number of dimensions in any one term, and that of the term itself.

433. \[ \tan 360 = 3 \tan 60 \]

For \( \tan 60 = \frac{\sin 60}{\cos 60} = \frac{\sqrt{3}}{1} = \sqrt{3} \)

\[ \therefore \tan 360 = (\sqrt{3})^3 = \sqrt{27} = 3 \sqrt{3} = 3 \tan 60. \]

434. Let \( a + mb \) be the mean arc,

\( a + (m-p) \) \( b \), \( a + (m+p) \) \( b \) be the extremes

\( a \) being first term, and \( b \) the common difference

Then \( \cos (a+mb-pb) = \cos (a+mb-pb) \)

\[ = \cos (a+mb) \cdot \cos (pb) + \sin (a+mb) \cdot \sin (pb) \]

and \( \cos (a+mb+pb) = \cos (a+mb) \cdot \cos (pb) - \sin (a+mb) \cdot \sin (pb) \)

\[ \therefore \cos (a+mb-pb) + \cos (a+mb+pb) = 2 \cos (pb) \cdot \cos (a+mb) \]

\[ \therefore \cos (a+mb-pb) + \cos (a+mb+pb) = \frac{2}{r} \cos (pb) \cdot \cos (a+mb) \]

\[ \therefore r:2 \cos (pb) \therefore \cos (a+mb): \cos (a+mb-pb) + \cos (a+mb+pb) \]

a result which indicates an error in the enunciation of the problem.
435. \[ \tan \left(45 - \frac{z}{2}\right) = \frac{1 - \tan \frac{z}{2}}{1 + \tan \frac{z}{2}} \]

and \[\tan z + \sec z = \frac{\sin z}{\cos z} + \frac{1}{\cos z} = \frac{\sin z + 1}{\cos z} \]

\[= \frac{2 \sin \frac{z}{2} \cos \frac{z}{2} + \cos \frac{z}{2}^2 + \sin \frac{z}{2}^2}{\cos \frac{z}{2} - \sin \frac{z}{2}} = \frac{(\cos \frac{z}{2} + \sin \frac{z}{2})^2}{\cos \frac{z}{2} - \sin \frac{z}{2}} \]

\[= \frac{\cos \frac{z}{2} + \sin \frac{z}{2}}{\cos \frac{z}{2} - \sin \frac{z}{2}} = \frac{1 + \tan \frac{z}{2}}{1 - \tan \frac{z}{2}} = \frac{1}{\tan^2 u} \]

Again, \[\tan z - \sec z = \frac{\sin z - 1}{\cos z} = \frac{2 \sin \frac{z}{2} \cos \frac{z}{2} - (\cos \frac{z}{2} + \sin \frac{z}{2})^2}{\cos \frac{z}{2} - \sin \frac{z}{2}} \]

\[= - \frac{(\cos \frac{z}{2} - \sin \frac{z}{2})^2}{\cos \frac{z}{2} - \sin \frac{z}{2}} = \frac{\cos \frac{z}{2} - \sin \frac{z}{2}}{\cos \frac{z}{2} + \sin \frac{z}{2}} \]

\[= - \frac{1 - \tan \frac{z}{2}}{1 + \tan \frac{z}{2}} = - \tan^3 u, \]

\[\sqrt{\tan z + \sec z} + \sqrt{\tan z - \sec z} = \frac{1}{\tan u} - \tan u \]

\[= \frac{1 - \tan^2 u}{\tan u} \]

\[= 2 \times \frac{1}{\tan 2u} \]

\[= 2 \times \cot 2u. \]

436. The terms of the equation being expanded, we have

\[
\sin B + \sin A \quad \cos B - \cos A \quad \sin B + \sin 2A \quad \cos B + \cos 2A \times \\
\sin B = \sin A \quad \cos B + \cos A \quad \sin B + \sin 2A \quad \cos B - \cos 2A \times \\
\sin B. 
\]
\[ \sin B = 2 \cos A \sin B + 2 \cos 2A \sin B = 0 \]

\[ 2 \cos 2A - 2 \cos A = -1 \]

\[ 2 \cos^2 A - 1 - \cos A = -\frac{1}{2} \]

\[ \cos^2 A - \frac{1}{2} \cos A = \frac{1}{4} \]

\[ \cos A = \frac{1 \pm \sqrt{5}}{4} \], whence, and by reference to the tables, the numerical values of \( A \) may easily be obtained, one of which is 72°.

437. \[ \tan 30 = \tan \frac{60}{2} \]

But \( \tan 60 = \frac{2 \tan 30}{1 - \tan^2 30} \) to radius = unity

and \( \tan 60 = \frac{\sin 60}{\cos 60} = \frac{\sqrt{3}}{1/2} = \sqrt{3} \)

\[ \tan^2 30 + \frac{2}{\sqrt{3}} \tan 30 = 1 \]

\[ \tan 30 = -\frac{1}{\sqrt{3}} \pm \frac{1}{\sqrt{3}} = -\frac{1 \pm 2}{\sqrt{3}} \]

\[ = \frac{1}{\sqrt{3}} \text{ or } -\frac{1}{\sqrt{3}} \left( = -\sqrt{3} \right) \text{ to radius } 1 \]

\[ \tan 30 = \frac{10000}{\sqrt{3}} \text{ or } -10000 \sqrt{3} \text{ to radius } 10000 \]

\[ = \frac{10000}{1.7320508} \text{ or } -10000 \times (1.7320508) \]

\[ = 5773.50 \ldots \text{ or } -17320.508 \ldots. \]

438. \[ \sin (A - B) = \frac{7}{2} = \sin 30^\circ \text{ or } \sin (\pi - 30) \]

or \( \sin (2\pi + 30), \&c. \)

and generally \( \sin (A - B) = \sin (2m\pi + 30) \) or \( \sin (\frac{3\pi + 11\pi}{m} - 30) \) where \( m \) is any number whatever.
ARITHMETIC OF SINES.

\[
A - B = 2m\pi + 30, \text{ or } 2m + 1.\pi - 30
\]

Also cos. \(A + B) = \frac{r}{2} = \cos. 60, \text{ or } = \cos. (2\pi - 60), \text{ &c.}

and generally cos. \(A + B) = \cos. (2m\pi - 60)

\[
\therefore A + B = 2n\pi - 60
\]

and \(A - B = 2m\pi + 30 \text{ or } 2m + 1.\pi - 30)

\[
\therefore A = (n+m)\pi - 15, \text{ or } = (n+m)\pi + 45 \text{ which have in-
\]

\[
B = (n-m)\pi - 45, \text{ or } = (n-m)\pi - 75 \text{ numerable values, since } n \text{ and } m \text{ may be any positive integral}
\]

numbers whatever.

Let \(n = 0 \text{ and } m = 0\)

Then \(A = -15^\circ, \text{ or } 45^\circ\)
\[B = -45^\circ, \text{ or } -75^\circ\]

\[\text{which are particular values of } A \text{ and } B.\]

439. Let it be required to find the sine of any arc between

\(45^\circ \text{ and } 90^\circ, \text{ as } 45 + a, \text{ where } a \text{ is less than } 45.\)

Then sin. \(45 + a) = \sin. (90 - 45 - a) = \cos. (45 - a)

which is known by the table.

Again cos. \(45 + a) = \cos. (90 - 45 + a) = \sin. (45 - a) which is

also known by the table.

\[\therefore \text{ the table exhibits the sines and cosines of every arc in the first quadrant.}\]

Suppose now, the arc to be between 90 and 180.

Then sin. \(90 + 45 + a) = \sin. (180 - 45 - a) = \sin. \]

\(45 - a) = \text{ which is known by the table.}

and cos. \(90 + 45 + a) = \cos. (180 - 45 - a) = - \cos. (45 - a) which is also known by the table.

Again, let the arc be between 180 and 270.

Then, sin. \(180 + 45 + a) = - \sin. (45 + a) = - \cos. (45 - a)

and cos. \(180 + 45 + a) = - \cos. (45 + a) = - \sin. (45 - a)

which are known by the table.

Again, let the arc be between 270 and 360.
Then \( \sin(270° + 45° + \alpha) = \sin(360° - 45° - \alpha) = -\sin(45° - \alpha) \)
and \( \cos(360° - 45° - \alpha) = \cdots \cdots = \cos(45° - \alpha) \)
which are known by the table.

\[
\therefore \text{the table exhibits the sines and cosines of any arcs whatever.}
\]

440. \[
\tan 30° = \frac{\sin 30°}{\cos 30°} = \frac{\frac{1}{2}}{\pm \sqrt{3}} = \frac{1}{\pm \sqrt{3}}
\]
But \( \tan 30° = \tan (2 \times 15°) = \frac{2 \tan 15°}{1 - \tan^2 15°} = \frac{1}{\pm \sqrt{3}} \)
\[
\therefore \sqrt{3} \cdot \tan 15° = 1 - \tan^2 15°
\]
\[
\therefore \tan^2 15° + 2 \sqrt{3} \tan 15° + 3 = 1 + 3 = 4
\]
\[
\therefore \tan 15° = \pm 2 \div \sqrt{3}, \text{to radius } = \text{unity.}
\]

441. \[
\sin (a + b) = \sin a \cdot \cos b + \cos a \cdot \sin b
\]
\[
\sin (a - b) = \sin a \cdot \cos b - \cos a \cdot \sin b
\]
\[
\therefore \sin (a + b) = 2 \sin a \cdot \cos b - \sin (a - b)
\]
Let \( b = 1 \) and let \( \sin 1' = m \), and \( \cos 1' = \sqrt{1 - m^2} = n \)
Then \( (a + 1') = 2 \sin a \cdot n - \sin (a - 1') \), in which formula, if for \( a \) we substitute \( 1', 2', 3', \&c. \) successively, we shall determine the sines of \( 2', 3', \&c. \) in terms of the given quantities \( m \) and \( n \).
Thus \( \sin 2' = 2nm \)
\[
\sin 3' = 2n \sin 2' - \sin 1' = 4n^2m - m = m \cdot (3 - 4n^2) = m \cdot (3 - 4m^2)
\]
\[
\sin 4' = 2nm \cdot (3 - 4m^2) - 2nm = 4nm \cdot (1 - 2m^2)
\]
\&c. See Woodhouse's Trigonometry.

442. Let \( a \) be the first term, \( b \), the common difference and
\( a + mb \) the mean term.

\[
a + (m - p) b \}
\text{the equidistant extremes.}
\]
\[
a + (m + p) b \}
\]
Then \( \sin (a + m + p.b) = \sin (a + mb) \cdot \cos pb + \cos (a + mb) \cdot \sin pb \)
\[
\sin(a + m - p.b) = \sin(a + mb). \cos pb - \cos(a + mb) \\
\sin pb \\
\therefore \frac{\sin(a + m + p.b)}{r} + \frac{\sin(a + m - p.b)}{r} = \frac{2\sin(a + mb)}{r} \times \frac{\cos pb}{r} \\
\therefore r : 2\cos pb :: \sin(a + mb) : \sin(a + m + p.b) + \sin(a + m - p.b) \text{ a result which shows an error in the enunciation.} \\
\text{To apply this proposition as required, let } r = 1 \\
\text{Then } \sin(a + m + p.b) + \sin(a + m - p.b) = 2\cos pb \times \sin(a + mb) \\
\therefore \sin(a + mb + pb) = 2\cos pb \times \sin(a + mb) - \sin(a + mb - pb) \\
\text{Hence, having given } \sin pb, \text{ or } \cos pb, \sin(a + mb) \text{ and } \\
\sin(a + mb - pb), \text{ we can always obtain } \sin(a + mb + pb); \text{ from which we get } \cos(a + mb + pb), \text{ and } \therefore \sec(a + mb + pb), \\
\tan(a + mb + pb), \cot(a + mb + pb), \&c. \text{ and } \therefore \text{ by substituting different numbers in succession for } a + mb + pb, \text{ we obtain the} \\
sines, \cosines, \&c. \text{ of every arc, which, properly arranged, will form the table.} \\
\text{Thus, let } a = 1', 2', 3', 4', \&c. \text{ and } b = 1' \\
m + p \text{ being also } 1', 2', 3', \&c. \\
\text{Then, by Woodhouse having found the value of } \sin 1' = m, \text{ and} \\
\text{thence } \cos 1' = n, \text{ by the form, we have} \\
\sin 2' = 2\cos 1' \times \sin 1' - 0 = 2mn \\
\sin 3' = 2n \times 2mn - \sin 1' = 4mn^2 - m \\
\&c. = \&c. \\
\text{Hence, we have the sines of all arcs, and } \therefore \text{ the cosines,} \\
tangents, \&c. \text{ Different methods, however, should in particular} \\
cases be resorted to, for which see Woodhouse's Trigonometry.}
ARITHMETIC OF SINES.

\[
\therefore \sin B + \sin C = (b + c), \quad \sin B = (b + c) \cdot \frac{\sin A}{a} \quad \text{and is}
\]
\[
\therefore \text{known.}
\]

But \( \sin B + \sin C = 2 \sin \frac{B + C}{2} \cdot \cos \frac{B - C}{2} \)

and \( \frac{B + C}{2} \) being \( = \frac{\pi}{2} \), \( \sin \frac{B + C}{2} = \cos \frac{A}{2} \)

\[
\therefore \cos \frac{B - C}{2} = \frac{b + c}{a}, \quad \sin \frac{A}{2} = \frac{b + c}{a} \quad \text{sin} \frac{A}{2}, \quad \text{and is} \quad \therefore \text{known.}
\]

\[
\therefore \frac{B - C}{2} \text{ is known and } m = n.
\]

Also \( \frac{B + C}{2} \) \( = \frac{\pi}{2} = n \)

\[
\therefore B = n + m \quad \text{and} \quad C = n - m \quad \text{are known}
\]

\[
\therefore b = \frac{a}{\sin A} \times \sin B \]

\[
\text{and} \quad c = \frac{a}{\sin A} \times \sin C \]

444. \quad \cos B = \frac{1}{2}, \quad \sin A = \frac{1}{2} \cdot \frac{a}{c}

But \( \cos B \) also \( = \frac{a^2 + c^2 - b^2}{2ac} \) (Woodhouse.)

\[
\therefore \frac{a}{2c} = \frac{a^2 + c^2 - b^2}{2ac}
\]

\[
\therefore \frac{a^2}{2} = \frac{a^2 + c^2 - b^2}{2ac}
\]

\[
\therefore c^2 = b^2
\]

and \( c = b \), or the triangle is isosceles.

445. Variable quantities in changing sign must become zero or infinite.

Hence the sine being positive from zero to 180, where from a finite quantity it becomes zero, must afterwards change sign, or become negative; it becomes 0 again for the arc \( = 360^\circ \), after which it again changes sign, &c. &c.
The cosine changes sign after the first quadrant, because it has then passed through zero. For the same reason it again changes sign after the third quadrant. The tangent changes sign after the first quadrant, because it has then passed through infinity. After the second quadrant it again changes sign, because it has then passed through zero. It becomes infinite also at the end of the third quadrant, and :: again changes sign, &c. :: the tangent changes its sign in every quadrant, taking them in order.

The secant, never passing through zero, and only becoming infinite at the end of the first and third quadrants, has changes of signs through those points only.

Since the secants, in the first and third quadrants, have different signs, the change having taken place in passing from the first quadrant to the second; :: sec. A differs in sign from sec. \((180 + A)\) supposing \(A\) less than \(90\). If, however, \(A\) be \(> 90\) and = \(90 + A'\).

Then sec. \(A = \text{sec. } (90 + A') = - \text{sec. } A'.\)

sec. \((290 + A') = + \text{sec. } A'\), which have different signs, both having changed signs. A similar mode of proof will explain the other cases.

A more simple manner of proving this problem, would have been to have considered the functions of the arcs as positive or negative with regard to their position, with respect to the diameter of the circle, or its centre.

\[446. \quad \cos C = \frac{a^2 + b^2 - c^2}{2ab} \quad (\text{Woodhouse.})\]

\[\therefore 2ab \cos C = a^2 + b^2 - c^2\]

\[\therefore c^2 = a^2 - 2ab \cos C + b^2\]

and \(c = \pm \sqrt{a^2 - 2ab \cos C + b^2}\)

and sin. \(A = \pm \frac{a \cdot \sin C}{c}\)

\[\therefore \text{sin. } A = \frac{a \cdot \sin C}{c} = \pm \frac{a \cdot \sin C}{\sqrt{a^2 - 2ab \cos C + b^2}}\]

\[447. \quad (\text{Fig. 52.}) \quad \text{Let } BCD \text{ be the horizontal plane, }\]

\(MC\) the mountain, \(M\) being its summit.
ARITHMETIC OF SINES.

Take any station A, and with the quadrant measure the $\angle M A m$, which put $= \theta$, retreat to another station B, in such a direction, that the plane $M B A$ be perpendicular to the horizon, and let the distance between A and B be measured, which put $= a$. Also measure the $\angle M B C = \theta'$.

Then, if $M m$ be perpendicular to the horizon, it lies in the same plane with $M A$, $M B$, $B A$, &c.

and we have $M m = A m \times \tan \theta = B m \times \tan \theta' = a \tan \theta' + A m \tan \theta'$.

\[A m = \frac{a \tan \theta'}{\tan \theta - \tan \theta'}\]

\[M m = \frac{a \times \tan \theta \times \tan \theta'}{\tan \theta - \tan \theta'} = \frac{a}{\cot \theta' - \cot \theta}\]

This method would be found in practice to lead to results very inaccurate, on account of the errors arising from the refraction of light, &c., &c. That these errors may be as small as possible, the stations should be taken such that $\theta$ and $\theta'$ may be nearly of the form $45^\circ + \varphi, 45^\circ - \varphi$.

If the summit of the mountain were accessible, and of considerable altitude, that altitude may be found by means of the barometer.

448. (Fig. 53.) Let $O o$ be the obelisk standing on the declivity $o C$ (which is supposed perfectly gradual,) and $\theta, \theta'$ be the observed $\angle$ of elevation, at the corresponding distances $A o$, $B o$. Call $A o = a$ and $B A = a'$.

Now $A o : A B \propto \sin \theta' \propto \sin B A O$

or $A o : a' \propto \sin \theta' \propto \sin (\theta - \theta')$

\[A o = \frac{a' \sin \theta'}{\sin (\theta - \theta')} = b\]

\[\frac{b}{a} = \frac{\sin y}{\sin x}\]

\[\therefore \frac{b}{a} - 1 = \frac{\sin y}{\sin x} - 1 \therefore \frac{b - a}{a} = \frac{\sin y - \sin x}{\sin x}\]
ARITHMETIC OF SINES.

\[ \frac{b}{a} + 1 = \frac{\sin y}{\sin x} + 1 \quad \therefore \quad \frac{b + a}{b - a} = \frac{\sin y + \sin x}{\sin y - \sin x} = \frac{2 \sin \frac{x + y}{2} \cos \frac{y - x}{2}}{2 \sin \frac{y - x}{2} \cos \frac{y + x}{2}} = \tan \frac{\frac{x + y}{2}}{\tan \frac{\frac{y - x}{2}}{2}}
\]

\[ \therefore \quad \tan \frac{\frac{y - x}{2}}{2} = \frac{b + a}{b - a} \times \tan \frac{\frac{x + y}{2}}{2} = \frac{b - a}{b + a} \cot \frac{\frac{y + x}{2}}{2}
\]

\[ \begin{align*}
\text{and } \frac{\frac{y - x}{2}}{2} & \text{ is known from the tables } \\
\text{and } \frac{\frac{x + y}{2}}{2} & \text{ is known } = \frac{\pi}{2} - \frac{\frac{y + x}{2}}{2} \\
\text{whence we have } y \text{ and } x.
\end{align*}
\]

Hence \(OO = a \times \frac{\sin \frac{\theta}{2}}{\sin x} \) is known.

Otherwise:

Suppose \(OO\) produced to meet the perpendicular \(AN\). Then, with the quadrant, measure the \(\angle oAN = \phi\) from the station \(A\)
Also measure the \(\angle OAN = \theta + \phi = \phi'\) by supposition.

Then \(\therefore \quad AO = a, \quad OA = a. \quad \sin \phi \text{ is known, and } \quad AN = a. \quad \cos \phi \text{ is known;}
\]

and \(\therefore \quad ON = AN \times \tan \phi' = a. \quad \cos \phi, \quad \tan \phi' \text{ is known.}
\]

\(\therefore \quad OO = ON - ON = a (\cos \phi, \tan \phi' - \sin \phi) \text{ is known.}
\]

449. (Fig. 54.) Let the inaccessible object \(M\), as a kite, cloud, &c., be distant from the horizon by the interval \(MN = x\).
Take three stations \(A, B, C\) in the same straight line, such that \(AB = BC = m\) a known distance; and at these stations let the respective \(\angle\) of elevation be \(a, b, c\).

Then, since \(MN\) is perpendicular to the horizon, \(\angle MNA, \angle MNB, \text{ and } \angle MNC\) are right angles. And, by letting fall
a straight line from M, perpendicular upon CA, it may easily be
shewn that $CM^2 + MA^2 = 2CB^2 + 2BM^2$

$$= 2m^2 + 2BN^2 + 2x^2$$
or $CN^2 + x^2 + AN^2 + x^2 = 2m^2 + 2BN^2 + 2x^2$

$$\therefore CN^2 + AN^2 = 2m^2 + 2BN^2$$

But $CN = x \cdot \cot c$

$BN = x \cdot \cot b$

$AN = x \cdot \cot a$

$$\therefore x^2 \cdot \cot^2 c + x^2 \cdot \cot^2 a = 2m^2 + 2x^2 \cdot \cot^2 b$$

$$\therefore x^2 = \frac{2m^2}{\cot^2 c + \cot^2 a - 2 \cot^2 b}$$

$$\therefore x = \pm \frac{m \sqrt{2}}{\sqrt{\cot^2 a + \cot^2 c - 2 \cot^2 b}}$$

The problem will not be difficult, if AB and BC be unequal, as m
and n, x in this case:

$= \pm \frac{(m + n) \times m n}{\sqrt{n \cot 2a + m \cot 2b - m + n \times \cot 2b}}$

That the errors arising from the observed $\angle$ may be the less,
the $\angle a, b, c$ should be as nearly $= 45^\circ$ as possible. Allow-
ance must also be made for the refraction of light, &c. &c.

To adapt the above form to logarithmic computation,

$$x^2 = \frac{2m^2 \tan^2 a}{1 + \tan^2 a \cdot (\cot^2 c - 2 \cot^2 b)} = \frac{2m^2 \tan^2 a}{1 + \tan^2 b}$$

$$= \frac{2m^2 \tan^2 a}{\sec^2 b} = 2m^2 \tan^2 a \times \cos^2 b$$, by assuming $\tan^2 b$

whence we have,

$2 \log (\tan b) = 2 \log (\tan a) + \log (\cot^2 c - 2 \cot^2 b)$

and $\therefore \cos b$ will be known from the tables.

Hence $2 \log x = \log 2 + 2 \log m + 2 \log (\tan a) +$

$2 \log (\cos b)$

$$\therefore x = \frac{2 \log 2}{2} + \log m + \log (\tan a) + \log (\cos b)$$

$$\therefore \log x = \text{is known from the tables}$$

whence $x$ may also be found.
ARITHMETIC OF SINES.

450. \[ \cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c} \] (Woodhouse.)

\[ = \frac{\cos a - \cos^2 b}{\sin^2 b} \]

\[ \therefore \sin 2b \cos A = \cos a - 1 + \sin 2b \]

\[ \therefore \sin 2b = \frac{1 - \cos a}{1 - \cos A} \]

But \[ \cos A = 1 - 2 \sin^2 \frac{A}{2} \]

\[ \therefore 1 - \cos A = 2 \sin^2 \frac{A}{2} \]

and similarly \[ 1 - \cos a = 2 \sin^2 \frac{a}{2} \]

\[ \therefore \sin 2b = \frac{2 \sin \frac{a}{2} \sin \frac{A}{2}}{2 \sin \frac{A}{2}} = \frac{\sin \frac{a}{2}}{2} = \frac{\sin \frac{2a}{2}}{2} \]

\[ \therefore \sin b = \frac{\sin \frac{A}{2}}{\sin \frac{a}{2}} \cdot \frac{\sin a}{\sin A} \times \frac{\sin \frac{A}{2}}{\sin \frac{A}{2}} \]

Again, \[ \sin B = \frac{\sin b \times \sin A}{\sin a} = \frac{\sin \frac{a}{2}}{\sin a} \times \frac{2 \sin \frac{A}{2} \cos \frac{A}{2}}{\sin \frac{A}{2}} \]

\[ = \cos \frac{A}{2} \]

\[ = \cos \frac{a}{2} \]

451. Let \( A \) and \( B \) be the two arcs.

Then \( \tan A + \tan B = \frac{\sin A}{\cos A} + \frac{\sin B}{\cos B} = \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B} \)
ARITHMETIC OF SINES.

\[ \tan A - \tan B = \frac{\sin A \cdot \sin B}{\cos A \cdot \cos B} = \frac{\sin A \cdot \cos B - \cos A \cdot \sin B}{\cos A \cdot \cos B} = \frac{\sin (A - B)}{\cos A \cdot \cos B} \]

\[ \therefore \tan A + \tan B : \tan A - \tan B :: \sin (A + B) : \sin (A - B) \]

whatever may be the radius, for every term is of the same number of dimensions.

452. \[ \tan A = r \times \frac{\sin A}{\cos A}, \quad r \text{ being the radius} \]

and \[ \cot A = r \times \frac{\cos A}{\sin A} = \frac{r}{\frac{\sin A}{\cos A}} = \frac{r}{\tan A} \]

\[ \therefore \cot A : r :: r \cdot \tan A \]

Also, \[ \sin A \times \cos A = \frac{\sin 2A}{2} \propto \sin 2A, \quad \text{(since 2 is invariable.)} \]

453. Let the \( \sin A = m. \)

Then \( \sin 2A = 2 \sin A \cos A = 2 \sin A \sqrt{1 - \sin^2 A} \)

\[ = 2m \times \sqrt{1 - m^2} \text{ which is known.} \]

454. \[ \therefore \sin (B + A) = \sin B \cos A + \cos B \sin A \]

\[ \sin (B - A) = \sin B \cos A - \cos B \sin A \]

\[ \therefore \sin (B + A) = 2 \cos A \sin B - \sin (B - A) \]

Let \( B = (n - 1) A, \text{ and substitute, &c.} \)

Then \( \sin nA = 2 \cos A \sin (n - 1) A - \sin (n - 2) A \)

Again, \[ \cos (B + A) = \cos B \cos A - \sin B \sin A \]

\[ \cos (B - A) = \cos B \cos A + \sin B \sin A \]
ARITHMETIC OF SINES.

\[ \therefore \cos (B + A) = 2 \cos A \times \cos B - \cos (B - A) \]

which if for \( B \) we substitute \( (n-1)A \), we get

\[ \cos nA = 2 \cos A \cos (n-1)A - \cos (n-2)A. \]

These forms being of great use in constructing tables, \&c. \&c. ought to be committed to memory.

455. (Fig. 55). Let \( CA \) be any radius of the given circle, and \( AB \perp \) to it. Take \( AB \) of any magnitude less than \( AC \), and divide in \( B \) so that

\[ \frac{Ab}{AB} = \frac{9}{4} \]

Draw \( BN, bn \perp Ab \) and meeting the circle in \( N, n \), and \( NM, nm \perp CA \).

Then \( \frac{nm}{NM} = \frac{Ab}{AB} = \frac{9}{4} \).

or \( \sin \angle nCA = \sin \angle NCA = \frac{9}{4} \).

\[ \therefore \angle nCA, NCA \text{ are such as we required to be found.} \]

It is evident that the problem admits of innumerable solutions.

456. \[ \frac{\cos A + \sin A}{\cos A - \sin A} = \frac{(\cos A + \sin A)^2}{\cos^2 A - \sin^2 A} \]

\[ = \frac{\cos^2 A + 2 \sin A \cos A + \sin^2 A}{\cos^2 A - \sin^2 A} = \frac{1 + \sin 2A}{\cos 2A} \]

\[ = \frac{1}{\cos 2A} + \frac{\sin 2A}{\cos 2A} = \sec 2A + \tan 2A. \]

Again, \( \tan B = \sec A - \tan A = \frac{1}{\cos A} - \frac{\sin A}{\cos A} \)

\[ = \frac{1 - \sin A}{\cos A} = \frac{\cos^2 A + \sin^2 A}{2 \cos A} - \frac{2 \sin A \cos A}{2 \cos^2 A} = \frac{\cos A}{\cos^2 A - \sin^2 A} \]

\[ = \frac{(\cos A - \sin A)^2}{\cos^2 A - \sin^2 A} = \frac{\cos A - \sin A}{\cos A + \sin A} \]

Divide both numerator and denominator, by \( \cos A \).
Then \( \tan B = \frac{1 - \tan \frac{A}{2}}{1 + \tan \frac{A}{2}} \)

But \( \tan \left( m\pi + 45 - \frac{A}{2} \right) = \frac{\tan (m\pi + 45) - \tan \frac{A}{2}}{1 + \tan (m\pi + 45) \times \tan \frac{A}{2}} \)

and \( \tan (m\pi + 45) = \frac{\tan m\pi + \tan 45}{1 - \tan m\pi \times \tan 45} \)

\[ = \frac{\tan 45}{1} = \frac{1}{1} = 1 \] (\( m \) being any integral number whatever.)

\[ \therefore \tan (m\pi + 45 - \frac{A}{2}) = \frac{1 - \tan \frac{A}{2}}{1 + \tan \frac{A}{2}} = \tan B \]

\[ \therefore B = m\pi + 45 - \frac{A}{2} \] which is a general solution (\( m \) being any integer whatever.)
DIFFERENTIAL CALCULUS.

4.19 × 457. Let \( \frac{x}{(a^2 - x^2)^{1/2}} = y \)

\[ \therefore x^2 = y^2 \times (a^2 - x^2) \]

\[ \therefore 2x \, dx = 2y \, dy \times (a^2 - x^2) - 2x \, dx \times y^2 \]

\[ \therefore dy = \frac{2x \, dx \times (1 + y^2)}{2y \times (a^2 - x^2)} \]

But \( 1 + y^2 = 1 + \frac{x^2}{a^2 - x^2} = \frac{a^2}{a^2 - x^2} \)

\[ \therefore d. \frac{x}{(a^2 - x^2)^{1/2}} = dy = \frac{2x \, dx \times \frac{a^2}{a^2 - x^2}}{(a^2 - x^2)^{3/2}} = \frac{a^2 \, dx}{(a^2 - x^2)^{3/2}} \]

Again, let \( \frac{\sqrt{a^2 + x^2}}{\sqrt{a^2 - x^2}} = z \)

Then \( a^2 + x^2 = z^2 \, a^2 - x^2 \)

\[ \therefore 2x \, dx = 2a^2 \, x \, dz - 2z \, dx \times x^2 = 2x \, dx \cdot z \]

\[ \therefore (a^2 \cdot z - z \cdot x^2) \, dz = (x + x^2) \, dx \]

\[ \therefore dz = \frac{x \cdot (1 + z^2) \, dx}{2 \times (a^2 - x^2)} \]

But \( 1 + z^2 = 1 + \frac{a^2 + x^2}{a^2 - x^2} = \frac{2a^2}{a^2 - x^2} \)

and \[ \frac{1 + z^2}{z} = \frac{2a^2}{a^2 - x^2} \times \frac{\sqrt{a^2 - x^2}}{\sqrt{a^2 + x^2}} = \frac{2a^2}{\sqrt{a^2 - x^2}} \]

\[ \therefore dz = \frac{2a^2 \, x \, dx}{(a^2 - x^2) \sqrt{a^2 - x^2}} \]
Again, since \( d \log u = M \frac{du}{u} \left( \frac{1}{M} \right) \) being the modulus. Put 

\[ a^2 = u \]

Then \( x \log a = \log u \)

\[ \therefore d \log u = M \times \frac{du}{u} = M \times \log a \times dx \]

\[ \therefore du = u \times \log a \times dx = \log a \times a^2 dx \quad \text{See Lacroix.} \]

This may be found independently of logarithms, by taking for the differential of the function, the second term of \( a^2 + dx \), developed according to the powers of \( dx \).

\[ 3.16 \quad 458. \quad \text{Let} \quad \sqrt[3]{a^3 + x^3 - \sqrt[3]{a^2 - x^2}} = u \]

Then \( a^3 + x^3 - \sqrt[3]{a^2 - x^2} = u^2 \)

\[ \therefore 3 x^2 dx + \frac{x dx}{\sqrt[3]{a^2 - x^2}} = 2u \frac{du}{dx} \]

\[ \therefore du = \frac{(3x \sqrt[3]{a^2 - x^2} + 1) xdx}{2 \sqrt[3]{a^2 - x^2} \sqrt[3]{a^3 + x^3 - \sqrt[3]{a^2 - x^2}}} \]

\[ 459. \quad \text{Let} \quad \frac{x}{\sqrt{1 + x^2}} = u \]

\[ \therefore x^2 = u^2 + u^2 \times x^2 \]

\[ \therefore 2x dx = 2u \frac{du}{dx} + 2udu \times x^2 + 2x dx \times u^2 \]

\[ \therefore udu \times (1 + x^2) = xdx \times (1 - u^2) \]

and \( du = \frac{x dx}{1 + x^2} \times \frac{1}{u} \)

But \( 1 - u^2 = 1 - \frac{x^2}{1 + x^2} = \frac{1}{1 + x^2} \)

\[ \therefore \frac{1 - u^2}{u} = \frac{1}{1 + x^2} \times \sqrt{1 + x^2} = \frac{1}{x \sqrt{1 + x^2}} \]

\[ \therefore du = \frac{dx}{(1 + x^2)^{\frac{3}{2}}} \]
Differential Calculus.

77.10

460. Let \( a + bx + cx^2 = u^2 \)
\[ \therefore bdx + 2cx dx = 2u du \]
\[ \therefore du = \frac{(b + 2cx) dx}{2 \sqrt{a + bx + cx^2}} \]

Again, put \( \frac{1}{\sqrt{a + x}} = v \) \( \therefore 1 = av^2 + x v^2 \)
\[ \therefore 2av dv + v^2 dx + 2vdv \times x \]
\[ \therefore 2vdv \times (a + x) = -v^2 dx \]
\[ \therefore dv = -\frac{dx \times v}{2(a + x)} = -\frac{dx}{2(a + x)^{\frac{3}{2}}} \]

Again, put \((a^2)^{\frac{3}{2}} = u\)
\[ \therefore x \log a^2 = \log u \]
\[ \therefore x^2 \log a = \log u \]
\[ \therefore 2 \log a \times x dx = \frac{du}{u} \times M_{\frac{1}{M}} \]

being the modulus.

\[ \therefore du = \frac{2 \log a}{M} x \times (a^2)^{\frac{3}{2}} dx \]

461. Let \( \frac{a}{\sqrt{x^2 + y}} = u \)
\[ \therefore a^2 = u^2 x^2 + u^2 y \]
\[ \therefore 0 = 2u du \times x^2 + 2x dx \times u^2 + 2u du \times y + u^2 dy \]
\[ \therefore 2u du (x^2 + y) = -u^2 (2x dx + dy) \]
\[ \therefore du = -\frac{u}{2} \left( \frac{2x dx + dy}{x^2 + y} \right) \]
\[ = -\frac{a}{2} \times \frac{2x dx + dy}{(x^2 + y)^{\frac{3}{2}}} \]

Again, put \( \frac{(x + a)^2}{\sqrt{x^2 - a^2}} = u \)

Then \( (x + a)^4 = u^2 x^2 - u^2 a^2 \)
\[ \therefore 4 (x + a)^3 dx = 2u du \times x^2 + 2x dx \times u^2 - 2udu \times a^2 \]
\[ \therefore udu \times (x^2 - a^2) = (2 (x + a)^3) - x \times u^2 \times dx \]

But \( 2 (x + a)^3 - x \times u^2 = 2 (x + a)^3 - \frac{x \times (x + a)^4}{x^2 - a^2} \)
\[
= (x+a)^3 \frac{(2x^2 - 2a^2 - x^2 - ax)}{x^2 - a^2} = (x+a)^3 \times \frac{(x^2 - a \cdot 2a + x)}{x^2 - a^2}
\]

\[
\therefore \ du = \sqrt{\frac{x^2 - a^2}{(x^2 - a^2)(x + a)^2}} \times \frac{(x+a)^3 \times (x^2 - a \times 2a + x)dx}{x^2 - a^2}
\]

\[
\therefore \ \frac{(x+a) \times (x^2 - a \cdot 2a + x)}{(x^2 - a^2)^{\frac{3}{2}}} \ \frac{dx}{(x^2 - a^2)^{\frac{3}{2}}} = \frac{(x+a)^3 \times (x - 2a)}{x^2 - 2a} \ \frac{dx}{(x^2 - a^2)^{\frac{3}{2}}}
\]

for \( x^2 - ax - 2a^2 = x^2 - a^2 - a, x + a = x + a, x - 2a \)

Otherwise.

\[
\log u = 2 \log (x + a) - \frac{1}{2} \log (x^2 - a^2)
\]

\[
\therefore \ du = 2 \frac{dx}{x + a} - \frac{xdx}{x^2 - a^2}
\]

\[
\therefore \ & c. \ & c. \ & c.
\]

462. Let \( l(\frac{1-x^3}{1+x^3}) = l(1 - x^3) - l(1 + x^3) = u \)

\[
l(1 + x^3) = u
\]

\[
\therefore \ du = -\frac{3}{2} x^\frac{1}{3} \ dx - \frac{3}{2} x^\frac{1}{3} \ dx
\]

\[
\therefore \ \frac{3}{2} x^\frac{1}{3} \ dx \left( \frac{1}{1-x^3} + \frac{1}{1+x^3} \right)
\]

\[
= -\frac{3}{2} x^\frac{1}{3} \ dx \times \frac{2}{1-x^3}
\]

\[
= \frac{3x^\frac{1}{3} \ dx}{x^3 - 1}
\]

\[
d. \ \frac{x}{1+x^3} = \frac{dx(1+x^3) - 2x^2 dx}{(1+x^3)^2} = \frac{1-x^2}{(1+x^3)^2} \ dx
\]

463. Let \( l. x e^{\cos x} = u \)
\[ du = \frac{dx}{x} e^{\cos x} + \frac{d}{dx} e^{\cos x} \times l. x \]

Let \( e^{\cos x} = v \)

\[ \therefore l. v = \cos x \]

\[ \therefore \frac{dv}{v} = -\sin x \times dx \]

\[ \therefore dv = -\sin x \times e^{\cos x} \times dx \]

\[ \therefore du = \frac{dx}{x} e^{\cos x} - \sin x e^{\cos x} \times dx \]

\[ = e^{\cos x} \times \left( \frac{1}{x} - \sin x \times l. x \right) \]

\[ = \frac{e^{\cos x}}{x} \times (1 - x \sin x \times l. x) \times dx \]

464. Let \( \frac{x}{\sqrt{1 + x^2}} = u \)

\[ \therefore \log u = \log x - \frac{1}{2} \log (1 + x^2) \]

\[ \therefore \frac{du}{u} = \frac{dx}{x} - \frac{xdx}{1 + x^2} = \frac{dx}{x \times (1 + x^2)} \]

\[ \therefore du = \frac{dx}{(1 + x^2)^{\frac{3}{2}}} \]

\[ \therefore \log u = \log x - \frac{1}{2} \log (1 + x^2) \]

465. \[ d \left( 1 - x^{\frac{1}{2}} + x^{\frac{3}{2}} \right) = \left( \frac{1}{2} x^{\frac{1}{2} - \frac{3}{2}} - \frac{3}{2} x^{\frac{1}{2}} \right) \]

\[ \frac{1}{3x^{\frac{3}{2}}} - \frac{1}{2x^{\frac{1}{2}}} = \frac{2x^{\frac{1}{2}} - 3x^{\frac{3}{2}}}{6x^3} = \frac{2 - 3x^{\frac{1}{2}}}{6x^3} \]

\[ \therefore \frac{1}{3x^{\frac{3}{2}}} - \frac{1}{2x^{\frac{1}{2}}} = \frac{2x^{\frac{1}{2}} - 3x^{\frac{3}{2}}}{6x^3} \]

\[ \therefore d \left( 1 - x^{\frac{1}{2}} + x^{\frac{3}{2}} \right) = \frac{2 dx}{15x^{\frac{3}{2}}} \]

\[ = \frac{2 - 3x^{\frac{1}{2}}}{6x^3} \]

\[ = \frac{2 - 3x^{\frac{1}{2}}}{6x^3 (1 - x^{\frac{1}{2}} + x^{\frac{3}{2}})} \times dx \]

\[ \frac{1}{15} \times \frac{2 - 3x^{\frac{1}{2}}}{x^{\frac{3}{2}} (1 - x^{\frac{1}{2}} + x^{\frac{3}{2}})} \times dx \]
147. 6

466. \[
\frac{dz}{1+z} = \frac{dz}{(1+z)^2} = \frac{dz}{(1+z)^2}
\]

151. 7

467. Let \(\left(\frac{a^2-y^2}{y}\right)^{\frac{1}{2}} = u\)

\[u = \frac{1}{2} (a^2 - y^2) - \frac{1}{2} y\]

\[\frac{du}{u} = \frac{-y dy}{a^2 - y^2} - \frac{1}{2} \frac{dy}{y} = \frac{-dy}{y} \left(\frac{a^2 - y^2}{y} \times (2y + a^2 - y^2)\right) = -\frac{dy}{y} \left(\frac{a^2 + y^2}{a^2 - y^2}\right)
\]

\[du = -\frac{dy \times (a^2 + y^2)}{y^2 \times (a^2 - y^2)^{\frac{1}{2}}}
\]

160. 11

468. Let \(ay^x = u\), and \(l\) be the characteristic of hyperbolic logarithms.

Then \(l \cdot a + x l \cdot y = l \cdot u\)

\[\frac{du}{u} = l \cdot y \times dx + x \cdot \frac{dy}{y}\]

\[du = ay^x \cdot l \cdot y \times dx + ax \cdot y^{x-1} \cdot dy
\]

Again let \(y = e^z\)

Then \(z = l \cdot y\)

\[du = ay^{x-1} (zy dx + x dy)
\]

161. 12

469. \[
\frac{dy}{\sqrt{1+y^2}} = \frac{dy}{\sqrt{1+y^2}} = \frac{dy}{\sqrt{1+y^2}}
\]

\[= \frac{dy}{(1+y^2)^{\frac{1}{2}}} = \frac{dy}{(1+y^2)^{\frac{1}{2}}}
\]
470. Let \( \frac{a^4 + x^4}{x} = u \)

\[ \therefore a^4 + x^4 = u^2 x^2 \]

\[ \therefore 4 x^3 \, dx = 2 x^2 u \, du + 2 u^2 \, x \, dx \]

\[ \therefore du = \frac{(2 x^2 - u^2) \, dx}{x \, u} \]

But \( 2 x^2 - u^2 = 2 x^2 - \frac{a^4 + x^4}{x^2} = \frac{x^4 - a^4}{x^2} \)

\[ \therefore du = \frac{x^4 - a^4}{x^2} \times \frac{x}{(a^4 + x^4)^{\frac{1}{2}}} \times dx = \frac{x^4 - a^4}{x^2 \cdot (x^4 + a^4)^{\frac{1}{2}}} \times dx \]

471. Let \( \frac{x}{\sqrt{a^3 + x^3}} = u \)

\[ \therefore l \cdot u = l \cdot x - \frac{1}{2} l(a^3 + a^3) \]

\[ \therefore \frac{du}{u} = \frac{dx}{x} - \frac{3}{2} \frac{x^2 \, dx}{a^3 + x^3} \]

\[ = \frac{dx}{2x \cdot (a^3 + x^3)} \times (2a^3 + 2x^3 - 3x^3) \]

\[ \therefore \frac{du}{u} = \frac{dx}{2 \cdot (a^3 + x^3)^{\frac{3}{2}}} \times (2a^3 - x^3) \]

472. \( \frac{\sqrt{ax - b}}{\sqrt{a - x}} = \frac{a^\frac{1}{2} \, dx}{2 \sqrt{x}} \times \sqrt{a - x} + \frac{dx \times (\sqrt{ax - b})}{\sqrt{a - x}} = 0 \)

\[ \therefore \frac{a^\frac{1}{2} \sqrt{a - x}}{2 \sqrt{x}} + \frac{\sqrt{ax - b}}{2 \times \sqrt{a - x}} = 0 \]

\[ \therefore \sqrt{a} \cdot (a - x) + \sqrt{x} \cdot (\sqrt{ax - b}) = 0 \]

\[ \therefore a^\frac{3}{2} - \sqrt{a} \cdot x + \sqrt{a} \cdot x - b \sqrt{x} = 0 \]

\[ \therefore b \sqrt{x} = a^\frac{3}{2} \]

\[ n \rightarrow \]
Differential Calculus.

\[ \therefore \sqrt[3]{x} = \frac{a^3}{b} \]

and \[ x = \frac{a^3}{b^3} \]

204 II

473. Let \( z^y = u \)

Then \( l.u = yzl.z \)

\[ \therefore \frac{du}{u} = dy.zl.z + dz.yl.z + \frac{dz}{z} yz \]

\[ = dy.zl.z + dz.yl.z + ydz \]

\[ \therefore du = z^{y+1}l.z \times dy + y z^y (l.z + 1) \ dx \]

Again, let \( x^z = v \)

Then \( l.v = yl.x \)

and \( l.l.v = zl.y + l.l.x \)

\[ \therefore \frac{dv}{l.l.v} = dz.l.y + \frac{dz}{l.x} + \frac{dy}{y} \]

\[ \therefore dv = x^z \times yl.x \]

\[ = x^{y-1} \times y^{-1} (xlxl.y \times dz + xzl.x \times dy + y dx), \]

which is expressed in terms of \( dz, dy, dx \), and functions of \( z, y, \) and \( x \).

213 II

474. Let \((a^2 + x^2) \times \sqrt{a^2 - x^2} = u \)

\[ \therefore l.u = l.(a^2 + x^2) + \frac{1}{2} l.(a^2 - x^2) \]

\[ \therefore \frac{du}{u} = \frac{2xdx}{a^2 + x^2} - \frac{x dx}{a^2 - x^2} = \frac{xdx}{a^2 - x^2} \times (3a^2 - 2x^2 - a^2 - x^2) \]

\[ = \frac{xdx}{a^2 - x^2} \times (a^2 - 3x^2) \]

\[ \therefore du = \frac{xdx (a^2 - 3x^2) \sqrt{a^2 - x^2}}{\sqrt{a^2 - x^2} \times x dx} = \frac{a^2 - 3x^2}{\sqrt{a^2 - x^2}} \times x dx \]
Differential Calculus.

222. 11  
\[ d \left( a + bx^{\frac{2}{3}} + cx^{\frac{3}{2}} \right)^{\frac{1}{2}} = \frac{1}{2} \frac{\left( \frac{2}{3} bx^{\frac{1}{3}} + \frac{3}{2} cx^{-\frac{1}{2}} \right) dx}{(a + bx^{\frac{2}{3}} + cx^{\frac{3}{2}})^{\frac{3}{2}}} \]
\[ = \frac{9 bx^{\frac{5}{3}} + 4c}{12 x^{\frac{3}{2}} (a + bx^{\frac{2}{3}} + cx^{\frac{3}{2}})^{\frac{3}{2}}} \times dx \]
Again, let \( u = \frac{\sqrt{a^2 + x^2}}{\sqrt{a^2 - x^2}} = \frac{1}{2} \sqrt{(a^2 + x^2) - \frac{1}{2} \sqrt{(a^2 - x^2)}} \)
\[ \therefore \ du = \frac{xdx}{a^2 + x^2} + \frac{xdx}{a^2 - x^2} \]
\[ = \frac{xdx}{a^2 - x^2} \times (a^2 - x^2 + a^2 + x^2) \]
\[ = \frac{2a^2}{a^2 - x^2} \times dx \]

244. 11  
\[ d \left( a + x \right) \sqrt{a - x} = \frac{dx}{2\sqrt{a - x}} \times (a + x) = \frac{\left( 2a - 2x - a - x \right) dx}{2\sqrt{a - x}} = \frac{a - 3x}{2\sqrt{a - x}} \times dx \]
For \( d \cdot a^x \), see Lacroix, or Vince.

255. 11  
Let \( (x^m + bx^n)^p = u \)
\[ \therefore x^m + bx^n = u^p \]
\[ \therefore mx^{m-1} \times nb \times x^{n-1} dx = \frac{1}{p} u^{\frac{p-1}{p}} du \]
\[ \therefore du = px^{m-1} \times (m + nb \times x^{n-m}) \times u^{\frac{p-1}{p}} \]
\[ \therefore du = px^{m-1} \times (x^m + bx^n)^{p-1} \times (m + nb \times x^{n-m}) dx \]
\[ = p \cdot x^{m-1} \times (1 + bx^{n-m})^{p-1} \times (m + nb \times x^{n-m}) \times dx \]

268. 11  
\[ d \left( a + cz^n \right)^m \times zp = mncz^{m-1} \times (a + cz^n)^{m-1} \times zpdz \]
\[ + dz \times (a + cz^n)^m \times p + dp \times (a + cz^n)^m \times z = pdz \times (a + cz^n)^{m-1} \times (mncz^n + a + cz^n) + dp \times (a + cz^n)^m \times z = p \cdot (a + cz^n)^{m-1} \times (a + mn + 1 \cdot c \cdot z) dz + z \cdot (a + cz^n)^m \cdot dp \].
Again, let \( x \sqrt{\frac{1 + x^2}{1 - x^2}} = u \)

\[
\therefore \frac{du}{u} = \frac{dx}{x} + \frac{x dx}{1 + x^2} + \frac{xdx}{1 - x^2}
\]

\[
= \frac{dx}{x} + \frac{2xdx}{1 - x^4}
\]

\[
= \frac{dx}{x} \frac{(1 - x^4) + 2x^2}{1 - x^4}
\]

\[
: \frac{du}{u} = \frac{1 - x^4 + 2x^2}{1 - x^4} \times \sqrt{\frac{1 + x^2}{1 - x^2}} \times dx
\]

\[
= \frac{x^4 - 2x^2 - 1}{\sqrt{x^4 - 1}} \times \frac{dx}{1 - x^2}
\]

479. \( \text{Let} \frac{(x + a)^2}{\sqrt{x^2 - a^2}} = u \)

\[
\therefore 2 l. (x + a) - \frac{1}{2} l. (x^2 - a^2) = l. u
\]

\[
: \frac{du}{u} = \frac{2dx}{x+a} - \frac{xdx}{x^2-a^2} = \frac{x-2a}{x^2-a^2} \times dx
\]

\[
: du = \frac{(x+a)\cdot(x-2a)}{(x^2-a^2)^{\frac{3}{2}}} \times dx
\]

480. \( \frac{a+x}{a^2+x^2} = \frac{dx}{(a^2+x^2)} \frac{(a^2+x^2) - 2x \cdot a}{(a^2+x^2)^2} \cdot \frac{(a^2+x^2)^2}{(a+x)^3} \times dx \)

\[
= \frac{(a^2 - 2ax - x^2)}{(a^2 + x^2)^2} \times dx
\]

Again, let \( (a^2 + x^2) \sqrt{a^2 - x^2} = u \)

\[
\therefore l. x + l. (a^2 + x^2) + \frac{1}{2} l. (a^2 - x^2) = l. u
\]
Differential Calculus.

\[
\frac{du}{u} = \frac{dx}{x} + \frac{2rdr}{a^2 + r^2} - \frac{rdr}{a^2 - r^2}
\]
\[
= \frac{dx}{x} + xdr \cdot \frac{2a^2 - 2x^2 - a^2 - x^2}{a^4 - x^4}
\]
\[
= \frac{dx}{x} + xdr \cdot \frac{a^2 - x^2}{a^4 - x^4}
\]
\[
= \frac{dx}{x, (a^4 - x^4)} \times (a^4 - x^4 + a^2 x^2 - 3x^4)
\]
\[
= \frac{a^4 + a^2 x^2 - 4x^4}{x, (a^4 - x^4)} \cdot dx
\]

Again \( d \sec x = \frac{1}{\cos x} \) = \(- d \cos x \)
\[
\frac{\sin x}{\cos^2 x} \frac{dx}{x} \quad \text{(Lacroix or Vince.)}
\]
\[
\frac{\sin x}{1 - \sin^2 x} \quad \text{or} \quad \frac{\sin x}{1 - \sin^2 x} \sec x \cdot dx
\]

481. Let \( u = x \times e^{\tan x} \)
\[
\therefore \quad l. u = l. x + \tan x \quad \text{(since } l. e^{\tan x} = \tan x \log e = \tan x)\)
\[
\therefore \quad \frac{du}{u} = \frac{dx}{x} + d. \tan x.
\]

But \( d. \tan x = d. \frac{\sin x}{\cos x} = \frac{\cos x \times \cos x + \sin x \times \sin x}{\cos^2 x} \)
\[
\therefore \quad dx = \frac{dx}{\cos^2 x} 
\]
\[
\therefore \quad du = u \times \frac{dx}{x} + u \frac{dx}{\cos^2 x}
\]
\[
= \left( e^{\tan x} + \frac{x e^{\tan x}}{\cos^2 x} \right) \cdot dx = \frac{e^{\tan x}}{\cos^2 x} \times (\cos^3 x + x) \cdot dx
\]

482. Let \( z \) be the arc.

Then \( \tan z = \sqrt{\frac{1-x}{1+x}} \)
DIFFERENTIAL CALCULUS.

But $d. \tan. z = \frac{dz}{\cos^2 z} = dz. \sec^2 z = dz. (1 + \tan^2 z)$

$= dz. (1 + \frac{1-x}{1+x}) = \frac{2dz}{1+x}$

$\therefore dz = \frac{1+x}{2} \times d. \tan. z = \frac{1+x}{2} \times d. \sqrt{\frac{1-x}{1+x}}$

Put $u = \frac{\sqrt{1-x}}{1+x}$

$\therefore \ln. u = \frac{1}{2} \ln. (1-x) - \frac{1}{2} \ln. (1+x)$

$\therefore \frac{du}{u} = \frac{1}{2} \frac{dx}{1-x} - \frac{1}{2} \frac{dx}{1+x} = \frac{-dx}{1-x^2}$

$\therefore du = \frac{-dx}{1-x^2} \times \sqrt{\frac{1-x}{1+x}}$

Hence $dz = \frac{-dx}{2(1-x)} \sqrt{\frac{1-x}{1+x}} = \frac{-dx}{2\sqrt{1-x^2}}$. 
\[ \int \frac{dx}{1+x} = \ln (1 + x) \]

\[ \therefore \int \frac{dx}{1+x} = \ln (1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \&c. \]

\[ \therefore \int \frac{dx}{x} \int \frac{dx}{1+x} = \int dx - \int \frac{2x}{2} + \int \frac{x^2}{3} + \&c. + C \]

\[ = x - \frac{x^2}{2} + \frac{x^3}{3} + \&c. \]

(since \( C = 0 \) when \( x = 0 \))

Let \( x = 1 \)

Then between \( x = 0 \) and \( x = 1 \),

\[ \int \frac{dx}{x} \int \frac{dx}{1+x} = 1 - \frac{1}{2^2} + \frac{1}{3^2} + \&c. \]

Again, \( \sin x = x \cdot (x^2 - \pi^2) \cdot (x + \pi) \cdot (x - 2\pi) \cdot (x + 2\pi) \ldots \&c. \) to infinity, because if \( \sin x = 0 \), and \( 0, \pi, -\pi \&c. \) being substituted for \( x \) satisfies the equation.

\[ \therefore \sin x = x \cdot (x^2 - \pi^2) \cdot (x^2 - 2\pi^2) \cdot (x^2 - 3\pi^2) \cdot \&c. \]

to infinity.

But \( \sin x \) also \( = x - \frac{x^3}{1.2.3} + \&c. \ldots \)

\[ \therefore (\pi^2 - \pi^2) \cdot (x^2 - 2\pi^2) \cdot (x^2 - 3\pi^2) \ldots \&c. = 1 - \frac{x^2}{1.2.3} + \&c. \]
\[ \therefore x^3, 2^x x^3, \text{ &c. are roots of the values of } x^3 \text{ of the equation} \]
\[ 1 + \frac{x}{1, 2, 3} \&c. \]

Now, by the theory of equations, the last coefficient divided by the last but one is the sum of the reciprocals of the roots,
\[ \therefore \frac{2Q^2}{3} = \frac{x^3}{6} = 1 + \frac{1}{2^x} + \frac{1}{3^x} + \&c. \]
\[ \therefore 2 \times \left( \frac{1}{2^x} + \frac{1}{4^x} + \&c. \right) = \frac{1}{2} \left( 1 + \frac{1}{2^x} + \frac{1}{3^x} + \&c. \right) \]
\[ = \frac{Q^2}{3} \]

\[ \therefore \frac{2Q^2}{3} - \frac{Q^2}{3} = 1 + \frac{1}{2^x} + \frac{1}{3^x} + \&c. - 2 \left( \frac{1}{2^x} + \frac{1}{4^x} + \&c. \right) \]
\[ = 1 - \frac{1}{2^x} + \frac{1}{3^x} - \&c. \]

\[ \int \frac{dx}{x} \int \frac{dx}{1+x} = \frac{2Q^2}{3} - \frac{Q^2}{3} = \frac{Q^2}{3} \text{ the limits of } x \]

being 0 and 1.

13-17

484. Let \( z^2 = x \)

Then \( \frac{n}{2} z^{n-1} \, dz = dx \)

and \( \frac{z^{n-1} \, dz}{\sqrt{a^2 + z^n}} = \frac{2 \, dx}{n \sqrt{x^2 + a^2}} \)

Again, put \( \sqrt{x^2 + a^2} = v \)

\[ \therefore x \, dx = v \, dv \]
\[ \therefore dx : dv :: v : x \]
\[ \therefore dx : dx + dv :: v : v + x \]
\[ \therefore \frac{dx}{v} = \frac{dx + dv}{x + v} = d. \, l. \, (x + v) \]
\[ = d. \, l. \, (x + \sqrt{x^2 + a^2}) \]
\[ \int \frac{x^{n-1}}{\sqrt{a^n + x^n}} \, dx = \frac{2}{n} \ln(x + \sqrt{x^n + a^n}) \]

Again, since \( dpq = pdq + qdp \)

\[ \int pdq = pq - \int qdp \]

\[ \int v^2 \, dx = v^2 x - \int xd \cdot v^2 \]

But \( dv^2 = 2vdv = 2l \cdot x \frac{dx}{x} \)

\[ \int v^2 dx = v^2 x - 2 \int v dx \]

Similarly, \( \int vdx = vx - \int v\, dx = vx - x \)

\[ \int v^2 dx = v^2 x - 2vx + 2x \]

\[ = x (v^2 - 2v + 2) \]

485. Let \( \int \frac{dx}{x^3 (a+x)^2} = u \), and assume \( \frac{1}{x^2(a+x)} = v \)

\[ dv = dx - (a+x)^{-1} \]

\[ = -x^{-2} \cdot (a+x)^{-1} \cdot dx - (a+x)^{-2} x^{-2} dx \]

\[ = -2x^{-3} \cdot (a+x)^{-2} dx - 3(a+x)^{-2} x^{-2} dx \]

\[ : v = -2a \cdot \frac{u}{a} - 3 \int (a+x)^{-2} x^{-3} \, dx \]

\[ : u = - \frac{v}{2a} - 3 \frac{u}{2a} \int (a+x)^{-2} x^{-2} \, dx \]

Again, put \( \frac{1}{x(a+x)} = v_1 \)

Then \( dv_1 = \frac{1}{a} \cdot x^{-1} \cdot (a+x)^{-1} \)

\[ = -x^{-2} \cdot (a+x)^{-1} \cdot dx - (a+x)^{-2} x^{-1} dx \]

\[ = -a \cdot x^{-2} \cdot (a+x)^{-2} \cdot dx - 2 \cdot (a+x)^{-2} x^{-1} dx \]

\[ : \int x^{-2} \cdot (a+x)^{-2} \, dx = \frac{-v_1}{a} - \frac{2}{a} \int (a+x)^{-2} x^{-1} \, dx \]

\[ : u = - \frac{v}{2a} + \frac{3v_1}{2a^2} + \frac{3}{a^2} \int (a+x)^{-2} x^{-1} \, dx \]

Now, to find \( \int (a+x)^{-2} x^{-1} \, dx \), put \( \frac{1}{a+x} = \frac{w}{a} \)
\[ \frac{-dx}{(a + x)^2} = \frac{dw}{a} \quad \text{and} \quad x = \frac{a}{w} - a = \frac{a}{w} \cdot \frac{1}{1-w} \]

\[ \int (a + x)^{-2} \cdot x^{-1} \, dx = \int \frac{-dw}{a} \cdot \frac{x}{a \cdot (1-w)} = \int \frac{-w \, dw}{a^2(1-w)} \]

\[ = -\frac{1}{a^2} \int dw + \frac{1}{a^2} \int \frac{dw}{1-w} = -\frac{1}{a^2} w - \frac{1}{a^2} \cdot l \cdot (1-w) = -\frac{1}{a^2} \cdot l \cdot \left( \frac{1}{a+x} \right) \]

\[ l \cdot \left( \frac{1-a}{a+x} \right) - \frac{1}{a^2} w = -\frac{1}{a^2} \cdot l \cdot \frac{x}{(a+x)} - \frac{1}{a^2} \cdot \frac{a}{(a+x)} \]

Hence, we get

\[ u = -\frac{v}{2a} + \frac{3v}{2a^2} + \frac{3}{a^2} \left( -\frac{1}{a^2} \cdot l \cdot \frac{x}{a+x} \right) \]

\[ -\frac{1}{a^2} \cdot \frac{a}{a+x} \]

\[ = \frac{-1}{2a \cdot x^2 \cdot (a+x)} + \frac{3}{2a^2} \cdot x \cdot (a+x) - \frac{3}{a^2} \cdot \frac{1}{a+x} \]

\[ l \cdot \frac{x}{a+x} = \frac{-a^3 + 3a^2 x - 6x^2}{2a^2 \cdot (a+x)} - \frac{3}{a^4} \cdot \frac{x}{(a+x)} \]

Otherwise,

Assume \( \frac{1}{x^3 \cdot (a+x)^2} = A + B + \frac{C}{x} + \frac{P}{(a+x)^2} + \frac{Q}{(a+x)} \)

which fractions being reduced to a common denominator, equate the numerators on both sides of the equation. Then let \( x = -a \), and we get the value of \( A \); let \( x = 0 \), and we get \( P \). Next, differentiate, divide by \( dx \), and repeat the operation, which will give \( B \) and \( Q \). Differentiate again, and let \( x = -a \). Thence we have \( C \). Having found \( A, B, C, \&c., \) the sum of the integrals of \( \frac{A \, dx}{x^3}, \frac{B \, dx}{x^2}, \&c., \) will be the integral required.

Secondly, put \( \frac{1}{1-v} = v \), \( \therefore \, y = 1 - \frac{1}{v} = \frac{v-1}{v} \)

\[ \therefore \, y^2 + 1 = \frac{2v^2 - 2v + 1}{v^2} \]

and \( \frac{dy}{(1-y)^2} = dv \)

\[ \int \frac{dy}{(1-y)^2} = \int \frac{vdv}{\sqrt{2v^2 - 2v + 1}} = \frac{1}{\sqrt{2}} \int \frac{vdv}{\sqrt{v^2 - v + \frac{1}{4}}} \]

Again, let \( v - \frac{1}{2} = z \), \( \therefore \, dv = dz \)
and \( v^3 - v + \frac{1}{4} = z^3 \)

and \( \therefore \sqrt{v^3 - v + \frac{1}{4}} = \sqrt{z^3 + \frac{1}{4}} \)

Hence

\[
\int \frac{dy}{(1-y)^2 \sqrt{1+y^2}} = \frac{1}{\sqrt{2}} \int \frac{(z + \frac{1}{2}) \, dz}{\sqrt{z^3 + \frac{1}{4}}}
\]

\[
= \frac{1}{\sqrt{2}} \int \frac{z \, dz}{\sqrt{z^3 + \frac{1}{4}}} + \frac{1}{2} \int \frac{\, dz}{\sqrt{z^3 + \frac{1}{4}}}
\]

\[
= \frac{1}{\sqrt{2}} \sqrt{z^3 + \frac{1}{4}} + \frac{1}{2} \log \left( z + \sqrt{z^3 + \frac{1}{4}} \right)
\]

But \( z^3 + \frac{1}{4} = v^3 - v + \frac{1}{4} = \frac{1}{(1-y)^3} - \frac{1}{1-y} + \frac{1}{2} \)

\[
= 2 - 2 \cdot \frac{1}{1-y} + \frac{(1-y)^2}{2(1-y)} = \frac{y^2 + 1}{2(1-y)}
\]

and \( z = v - \frac{1}{2} = \frac{1}{1-y} - \frac{2}{2(1-y)} = \frac{1+y}{2(1-y)} \)

\[
\therefore \int \frac{dy}{(1-y)^2 \sqrt{1+y}} = \frac{1}{\sqrt{2}} \sqrt{1+y} + \frac{1}{2} \log \left( \frac{1+y + \sqrt{1+y^3}}{2(1-y)} \right)
\]

We learn from the process, that one assumption, viz.,

\( z = \frac{1+y}{2(1-y)} \) would have been sufficient.

Thirdly, let \( v = \frac{1}{x} \)

Then \( dv = - \frac{dx}{x^2} \)

\[
\therefore \sqrt{a^2 - ax + x^2} = \sqrt{a^2 - \frac{a}{v} + \frac{1}{v^2}} = \frac{\sqrt{a^2 v^2 - av + 1}}{v}
\]

\[
\int \frac{a \, dx}{x \sqrt{a^2 - ax + x^2}} = \int \frac{-a \, dv}{\sqrt{a^2 v^2 - av + 1}}
\]

Again, put \( av - \frac{1}{2} = w \)

Then \( a^2 v^2 - av + \frac{1}{4} = w^2 \)

\[
\therefore \sqrt{a^2 v^2 - av + 1} = \sqrt{w^2 + \frac{8}{4}}
\]

Also \( \frac{-a \, dv}{w} = - \frac{dw}{w} \)
\[
\therefore \int \frac{adx}{x \sqrt{a^2 - ax + x^2}} = \int \frac{-dw}{\sqrt{w^2 + \frac{3}{4}}} = -l \cdot (w + \sqrt{w^2 + \frac{3}{4}})
\]

But \( w = av - \frac{1}{2} = \frac{a}{x} - \frac{1}{2} = \frac{2a - x}{2x} \)

\[
\therefore w^2 + \frac{3}{4} = \frac{4a^2 - 4ax + x^2}{4x^2} + \frac{3}{4} = \frac{a^2 - ax + x^2}{x^2}
\]

\[
\therefore \int \frac{adx}{x \sqrt{a^2 - ax + x^2}} = -l \cdot \frac{2a - x + 2\sqrt{a^2 - ax + x^2}}{2x}
\]

2 0. 8 486. Let \( y = \frac{1}{x} \therefore x = \frac{1}{y} \)

\[
\therefore dx = -\frac{dy}{y^2}
\]

\[
\therefore \frac{dx}{x} = -\frac{dy}{y}
\]

Also \( a^2 - x^2 = a^2 - \frac{1}{y^2} = a^2y^2 - 1 \)

\[
\therefore \int \frac{d\cdot dx}{x \cdot (a^2 - x^2)} = -\frac{d}{2a^2} \int \frac{ydy}{y^2 - 1} = -\frac{d}{2a^2} l \cdot (a^2y^2 - 1)
\]

\[
= -\frac{d}{2a^2} \cdot l \cdot \frac{a^2 - x^2}{x^2} = \frac{d}{a^2} l \cdot x - \frac{d}{2a^2} l \cdot (a^2 - x^2)
\]

Again, let \( \frac{1}{(a+y)^{\frac{3}{2}}} = u \)

\[
\therefore -\frac{\frac{1}{2} dy}{(a+y)^{\frac{3}{2}}} = du, \text{ and } y = \frac{1}{u^2} - a = \frac{1-a\cdot u^2}{u^2}
\]

\[
\therefore \frac{hd\cdot y}{y \cdot (a+y)^{\frac{3}{2}}} = -\frac{2hdu}{u} = \frac{2h \cdot u}{au^2 - 1} = \frac{2h}{a} du + \frac{2hdu}{a(au^2 - 1)}
\]

by division.

\[
\therefore \int \frac{hd\cdot y}{y \cdot (a+y)^{\frac{3}{2}}} = \frac{2h}{a} u + \frac{2h}{a^2} \times \int \frac{du}{u^2 - \frac{1}{a}}
\]

Put \( \frac{1}{u^2 - \frac{1}{a}} = \frac{A}{u + \sqrt{\frac{1}{a}}} + \frac{B}{u - \sqrt{\frac{1}{a}}} \) (since the factors of

\( u^2 - \frac{1}{a} \) are \( u + \sqrt{\frac{1}{a}} \) and \( u - \sqrt{\frac{1}{a}} \))
\[ A + B \cdot u + (B - A) \cdot \sqrt{\frac{1}{a}} = 1 \]

\[ A + B = 0 \]

\[ A - B = -\sqrt{a} \]

\[ A = -\frac{\sqrt{a}}{2} \quad \text{and} \quad B = \frac{\sqrt{a}}{2} \]

\[ \int \frac{du}{u^2 - \frac{1}{a}} = \int \frac{\sqrt{a} \cdot du}{u - \sqrt{\frac{1}{a}}} - \int \frac{\sqrt{a} \cdot du}{u + \sqrt{\frac{1}{a}}} = \frac{\sqrt{a}}{2} \cdot \int \frac{u - \sqrt{\frac{1}{a}}}{u + \sqrt{\frac{1}{a}}}
\]

\[ \int \frac{hdy}{y \cdot (a + y)^{\frac{3}{2}}} = \frac{2h}{a} \cdot \frac{u + \frac{h}{a^2} - \frac{\sqrt{u}}{a}}{u + \frac{\sqrt{1/a}}{a}} \cdot l. \quad \text{which, by substitution, may easily be expressed in terms of } y. \]

\[ z^2 + 2az + 1 = \frac{ud^2 + 1 - a^2}{u^2 + r^2}, \quad \text{by supposition.} \]

\[ \text{Also } z^2 = u^2 - 2au + a^2 \]

\[ \therefore \quad zdz = udu - adu \]

\[ \frac{zdz}{1 + 2az + z^2} = \frac{udu - adu}{u^2 + r^2} = \frac{udu}{u^2 + r^2} - \frac{adu}{u^2 + r^2} \]

\[ = \frac{udu}{r^2 + 1} - \frac{a}{r} \times \frac{du}{u^2 + 1} \]

\[ \therefore \quad \int \frac{zdz}{1 + 2az + z^2} = \frac{1}{2} \cdot l \cdot \left( \frac{u^2}{r^2} + 1 \right) - \frac{a}{r} \cdot \tan^{-1} \frac{u}{r} \]

If \( a \) be > unity

\[ 1 - a^2 \text{ is negative } = -r^2, \quad \text{and the integral will be} \]

\[ \therefore \quad \int \frac{zdz}{1 + 2az + z^2} = \frac{1}{2} \cdot l \cdot \left( 1 - \frac{u^2}{r^2} \right) - \int \frac{adu}{u^2 - r^2} \]

Let \[ \frac{1}{u^2 - r^2} = \frac{A}{u + r} + \frac{B}{u - r} \]
\[
\therefore \frac{A+B}{u} + (B - A) r = 1
\]

or \( A + B = 0 \)

and \( B - A = \frac{1}{r} \)

\[
\therefore A = -\frac{1}{2r} \text{ and } B = \frac{1}{2r}
\]

\[
\therefore \int \frac{a}{u^2 - r^2} du = -\int \frac{2r}{u^2} + \int \frac{a}{u^2} \cdot \frac{u - r}{u + r}
\]

\[
\therefore \text{ the integral in this case } = \frac{1}{2} l \cdot (1 - \frac{u^2}{r^2}) - \frac{a}{2r} \cdot \frac{u - r}{u + r}
\]

See Demoivre's Miscellanea Analytica, Lib. III.

Again, put \( a^2 + x^2 = u^2 \)

\[
\therefore xdx = u du
\]

\[
\int \frac{x}{\sqrt{a^2 + x^2}} = \int \frac{u}{u} = f du = u = \sqrt{a^2 + x^2}
\]

488. Let \((a + cz^n)^m \times z^{m+1} \times dz = dP \)

and \((a + cz^n)^{m+1} \times z^{m-1} \times dz = dQ \)

Assume \((a + cz^n)^{m+1} \times z^n = u \)

\[
\therefore du = (m+1) n c (a + cz^n)^m z^{m+1} \times dz + pn(a + cz^n)^{m+1} \times z^{m-1} \times dz = m. m + 1. c dP + np. dQ
\]

\[
\therefore Q = \frac{1}{np} du - \frac{m + 1}{p} c dP
\]

\[
\therefore \frac{u}{np} \cdot \frac{p}{c P} = \frac{(a + cz^n)^{m+1} \times z^n}{np} - \frac{m + 1}{p} c P, \text{ whence }\]

\( Q \) is known.

Again, put \( \sqrt{a^2 + x^2} = u \)

\[
\therefore du = \frac{x^3 dx}{x^4 \sqrt{a^2 + x^2}} - 2xdx \cdot \frac{a}{x^2 \sqrt{a^2 + x^2}}
\]

\[
\therefore \int \frac{dx}{x^3 \sqrt{a^2 + x^2}} = \frac{1}{2} \int \frac{dx}{x \sqrt{a^2 + x^2}} - \frac{u}{2}
\]
INTEGRAL CALCULUS.

But \[ \int \frac{2adx}{x \sqrt{a^2 + x^2}} = \ln \frac{\sqrt{a^2 + x^2} - a}{\sqrt{a^2 + x^2} + a} \quad \text{(Vince or Lacroix)} \]

\[ \int \frac{dx}{x^3 \sqrt{a^2 + x^2}} = \frac{1}{4a} \ln \frac{\sqrt{a^2 + x^2} - a}{\sqrt{a^2 + x^2} + a} - \frac{u}{2} \]

\[ = \frac{1}{4a} \ln (\sqrt{a^2 + x^2} - a) - \frac{u}{2} \]

\[ = \frac{1}{2a} \ln \frac{\sqrt{a^2 + x^2} - a}{x} - \frac{\sqrt{a^2 + x^2}}{2x^3} \]

Again, \[ \frac{z^\theta}{1 + mz} = \frac{z^\theta}{mz + 1} = \frac{z^{\theta - 1}}{m} - \frac{z^{\theta - 2}}{m^2} + \frac{z^{\theta - 3}}{m^3} \]

\[ \text{&c.} \pm \frac{1}{m^\theta (1 + mz)} \quad \text{by actual division.} \]

\[ \int \frac{z^\theta dz}{1 + mz} = \int \frac{z^{\theta - 1}}{m} dz \int \frac{z^{\theta - 2}}{m^2} dz + \text{&c.} \pm \int \frac{dz}{m^\theta} \]

\[ \int \frac{dz}{m^\theta (1 + mz)} \]

\[ = \frac{z^\theta}{m^\theta (\theta + 1)} + \frac{z^{\theta - 2}}{m^\theta (\theta - 2)} + \frac{z^{\theta - 3}}{m^\theta (\theta - 3)} \pm \frac{z}{m^\theta} \]

\[ \int \frac{dz}{m^\theta + 1 (1 + mz)} \]

488. Since \[ \int udw = uv - \int vdw \]

\[ \int vxdx = v \times \frac{x^2}{2} - \int \frac{x^2}{2} dv \]

\[ = v \times \frac{x^2}{2} - \int \frac{x^2}{2} \times \frac{dx}{\sqrt{x^2 + a^2}} \]

Again put \[ x \sqrt{x^2 + a^2} = y \]

\[ dy = dx \sqrt{x^2 + a^2} + \frac{x^2 dx}{\sqrt{x^2 + a^2}} \]

\[ = \frac{2x^2 dx}{\sqrt{x^2 + a^2}} + \frac{a^2 dx}{\sqrt{x^2 + a^2}} \]

\[ \text{VOL. I.} \]
\[
\frac{x^2}{2 \sqrt{x^2 + a^2}} = \frac{dy}{4} - \frac{a^2}{4} \times \frac{dx}{\sqrt{x^2 + a^2}}
\]

Hence \( \int v \cdot x \, dx = \frac{vx^2}{2} - \frac{y}{4} + \frac{a^2}{4} v \)

\[= \frac{vx^2}{2} - \frac{x \sqrt{x^2 + a^2}}{4} + \frac{a^2}{4} v \]

\[= \frac{1}{4} \left( v \cdot \frac{2x^3}{a - x} - x \sqrt{x^2 + a^2} \right) \]

\[\therefore \text{ Let } x \sqrt{a - x} = u \]

\[\therefore \frac{dx}{\sqrt{a - x}} = \frac{x \, dx}{2 \sqrt{a - x}} = du \]

\[\therefore \frac{x \, dx}{\sqrt{a - x}} = 2dx \sqrt{a - x} - 2 \, du \]

\[\therefore \int \frac{x \, dx}{\sqrt{a - x}} = - \frac{4}{3} (a - x)^{3/2} - 2u \]

\[= - \frac{2}{3} \sqrt{a - x} \left( \frac{2}{3} a - \frac{2}{3} x + x \right) \]

\[= - \frac{2}{3} \sqrt{a - x} \left( \frac{2}{3} a + \frac{1}{3} x \right) \]

\[= - \frac{2}{3} \sqrt{a - x} (2a + x) \]

Again, \( \frac{x^2 \, dx}{a - x} = - \frac{x^3 \, dx}{x - a} = -dx \left( x + a + \frac{a^2}{x - a} \right) \)

\[= -dx \left( \frac{a^2}{x - a} \right) \]

\[\therefore \int \frac{x^2 \, dx}{a - x} = - \frac{x^3}{2} - ax + a^2 \cdot l. (a - x) \]
33.16

490. Let \( x^{-1}, \sqrt{a-x} = u \)

\[
\therefore \frac{dx}{x^2} = \int \frac{dx}{\sqrt{a-x}} = \int \frac{dx}{x} + \frac{dx}{2x\sqrt{a-x}} = \frac{dx}{2x\sqrt{a-x}}
\]

\[
\therefore \int \frac{dx}{x^2\sqrt{a-x}} = \frac{1}{2a} \int \frac{dx}{x\sqrt{a-x}} - \frac{u}{a}
\]

Let now \( a - x = z^2 \)

\[
\therefore dx = -2xdz
\]

\[
\therefore \frac{1}{x} = \frac{1}{a-z^2}
\]

\[
\therefore \int \frac{dx}{x\sqrt{a-x}} = \int \frac{-2xdz}{z(a-z^2)} = \int \frac{-2dz}{(a-z^2)}
\]

\[
\therefore \int \frac{dx}{x^3\sqrt{a-x}} = \int \frac{1}{z\sqrt{a+x} + \sqrt{a}} \frac{1}{a}\frac{z}{x}\frac{dx}{a-x}
\]

Again, \( y^{2n} = u \)

\[
\therefore \frac{n}{y} \frac{y^{n-1}}{dy} = \frac{u}{dy}
\]

\[
\therefore \frac{3}{y} \frac{y^{n-1}}{dy} = \frac{2}{n} \frac{u}{dy}
\]

And \( \sqrt{a^n-y^n} = \sqrt{a^n-u^n} = \sqrt{b^n-u^n} \) (by putting \( b^n = u^n \))

\[
\therefore \int y^{n-1} \frac{dy}{\sqrt{a^n-y^n}} = \frac{2}{n} \int u^n \frac{du}{\sqrt{b^n-u^n}}
\]

Let \( u (b^n - u^n)^{\frac{3}{2}} = w \)

Then \( du \times (b^n - u^n)^{\frac{3}{2}} = 3 (b^n - u^n)^{\frac{3}{2}} w^udu = dw \)

or \( b^n du (b^n - u^n)^{\frac{1}{2}} = 4 (b^n - u^n)^{\frac{1}{2}} w^udu = dw \)

\[
\therefore \int u^n \frac{du}{\sqrt{b^n-u^n}} = \frac{b^n}{4} \int du \frac{u^n}{\sqrt{b^n-u^n}} - \frac{w}{4}
\]
INTEGRAL CALCULUS.

But if $b$ be the radius of a circle, and $u$ the abscissa measured from the centre ($u$ being supposed less than $b$), then $\sqrt{b^2 - u^2} = \text{the corresponding ordinate}$, and $\int du \sqrt{b^2 - u^2} = \text{that part of the quadrant comprised between the radius } \perp \text{ line of abscissae,}$ and the ordinate $\sqrt{b^2 - u^2}$; which being put $= A$, we have

$$\int y^{\frac{3n}{2}} \, dy \sqrt{a^2 - y^n} = \frac{2}{n} \times \frac{b^2}{4} A - \frac{2}{n} \times \frac{w}{4} = \frac{b^2 A}{2n}$$

$$- \frac{u(b^2 - u^2)^{\frac{3}{2}}}{2n} = \frac{1}{2n} \left\{ a^n \times A - y^{\frac{n}{2}} \left( a^n - y^n \right)^{\frac{3}{2}} \right\}$$

Again, put $v = a \frac{1 - u}{1 + u}$

Then $dv = a \times \frac{-du}{(1 + u)^2} = \frac{-2adu}{(1 + u)^2}$

and $v + u = a \frac{1 - u + 1 + u}{1 + u} = a \frac{2a}{1 + u}$

$$\therefore \frac{dv}{v + a} = \frac{-du}{1 + u}$$

But $v^2 + a^2 = \frac{4a^2}{(1 + u)^2} - 2av = \frac{4a^2}{(1 + u)^2} - 2a^2 \frac{1 - u}{1 + u}$

$$= \frac{4a^2 - 2a^2 + 2a^2 u^2}{(1 + u)^2} = \frac{2a^2 (1 + u^2)}{(1 + u)^2}$$

$$\therefore \frac{vdu}{(a + v)(a^2 + v^2)} = a \frac{1 - u}{1 + u} \times \frac{-du}{1 + u} \times \frac{(1 + u)^2}{2a^2 \times (1 + u^2)}$$

$$= - \frac{1}{2a} \times \frac{(1 - u)du}{(1 + u)^2} = \frac{udu}{2a(1 + u^2)} - \frac{udu}{2a(1 + u^2)}$$

$$\therefore \int \frac{vdu}{(a + v)(a^2 + v^2)} = \frac{1}{4a} l. \frac{1 - u}{1 + u} - \frac{1}{2a} \tan^{-1} u$$

$$= \frac{1}{4a} l. \frac{2(a^2 + v^2)}{(a + v)^2} - \frac{1}{2a} \tan^{-1} \frac{a - v}{a + v}$$

$$= \frac{1}{4a} l. \frac{2(a^2 + v^2)}{(a + v)^2} - \frac{1}{2a} l. (a + v) - \frac{1}{2a} \tan^{-1} \frac{a - v}{a + v}$$

Otherwise.

Assume $\frac{1}{(a + v)(a^2 + v^2)} = \frac{A + Bu}{a^2 + v^2} + \frac{C}{a + v}$ reduce to a