\[
\frac{2a - b}{1 \cdot 2} x + \frac{3a - 2b}{2 \cdot 3} x^2 + \ldots. \infty = (ax - b)s + b
\]

Let \(2a - b = 3\)
And \(a - b = 1\) the common difference of the numerator.
Hence \(a = 2\), and \(b = 1\), and
\[
\frac{3}{1 \cdot 2} x + \frac{4}{2 \cdot 3} x^2 + \ldots. \infty = (2x - 1)s + 1
\]
Let \(x = \frac{1}{2}\)

Then \(\frac{3}{1 \cdot 2 \cdot 2} + \frac{4}{2 \cdot 3 \cdot 2^2} + \ldots. \infty = 0 + 1 = 1\).

To sum 1. 2 + 2. 3 + \ldots n \cdot (n + 1), see 685.

790. To sum \(1.24 + 2.35 + \ldots n \cdot (n + 1) \times (n + 3) = S\).
\[
\Delta S = (n + 1) \cdot (n + 2) \cdot (n + 4)
\]
\[
= n \cdot (n + 1) \cdot (n + 2) + 4(n + 1) \cdot (n + 2)
\]
\[
\therefore S = \frac{(n - 1) \cdot (n + 1) \cdot (n + 2) + 4n \cdot (n + 1) \cdot (n + 2) + C}{3}
\]
\[
= \frac{3n^2 + 13}{12} \times n \cdot (n + 1) \cdot (n + 2).
\]

To sum 1 + \(\frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 9} + \frac{1}{7 \cdot 13} + \ldots. \infty\).

Since \(dx + x^2dx + x^4dx + \ldots. \infty = \frac{dx}{1 - x^2}\)
\[
x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots. \infty = \int \frac{dx}{1 - x^2} = \frac{1}{2} \ln \frac{1 + x}{1 - x}.
\]
Also \(x + \frac{x^5}{5} + \frac{x^9}{9} + \ldots. \infty = \int \frac{dx}{1 - x^4} = \frac{1}{2} \int \frac{dx}{1 - x^2} + \frac{1}{2} \cdot \tan^{-1} x.
\]
Let \(x = 1\), and multiply the latter series by 2; then we have
\[2r2\]
\[
2 + \frac{2}{5} + \frac{2}{9} + \ldots \infty = \frac{1}{2} \cdot \frac{2}{0} + \frac{\pi}{4} \\
\text{and } 1 + \frac{1}{5} + \frac{1}{9} + \ldots \infty = \frac{1}{2} \cdot \frac{2}{0}
\]

Whence, by subtraction,

\[
1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{9} + \ldots \infty = \frac{\pi}{4}
\]

791. Given \( 1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots \infty = \frac{\pi^2}{6} \) (see 684) to find the sum of \( \frac{1}{1^2} \cdot \frac{1}{2} + \frac{1}{2^2} \cdot \frac{1}{3} + \ldots \infty = S. \)

The general term \( \frac{1}{n^2 \cdot (n + 1)} \) may be decomposed by the usual methods into

\[
\frac{1}{n^2} - \frac{1}{n} + \frac{1}{n + 1}
\]

\[
\therefore S = (1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots \infty) - \left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots\right) + \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots\right) = \frac{\pi^2}{6} - 1. \quad \text{(See 700)}.
\]

792. To sum \( \frac{5}{1 \cdot 2 \cdot 3} + \frac{6}{2 \cdot 3 \cdot 4} + \ldots \frac{n + 4}{n \cdot (n + 1) \cdot (n + 2)} = S. \)

\[
\Delta S = \frac{n + 5}{(n + 1) \cdot (n + 2) \cdot (n + 3)} = \frac{1}{(n + 1) \cdot (n + 2)} + \frac{2}{(n + 1) \cdot (n + 2) \cdot (n + 3)}
\]

\[
\therefore S = C - \frac{1}{n + 1} - \frac{1}{(n + 1) \cdot (n + 2)} = \frac{3}{2} - \frac{n + 3}{(n + 1) \cdot (n + 2)}
\]
INVERSE METHOD OF SERIES.

To sum \[ \frac{6^2}{1 \cdot 2 \cdot 3 \cdot 4} + \ldots + \frac{(n+5)^2}{n(n+1)(n+2)(n+3)(n+4)} = S. \]

\[ \Delta S = \frac{(n+6)^2}{(n+1)(n+2)(n+3)(n+4)} \]

\[ = \frac{(n+3)(n+4) + 5(n+4) + 4}{(n+1)(n+2)(n+3)(n+4)} = \frac{1}{(n+1)(n+2)} \]

\[ + \frac{5}{(n+1)(n+2)(n+3)} + \frac{4}{(n+1)\ldots(n+4)} \]

\[ \therefore S = C - \left( \frac{1}{n+1} + \frac{5}{2(n+1)(n+2)} + \frac{4}{3(n+1)(n+2)(n+3)} \right) \]

\[ = \frac{89}{36} - \frac{6n^2 + 45n + 89}{6(n+1)(n+2)(n+3)}. \]
PROBABILITIES.

793. The probability of failing both throws with the single die, is

\[ P = \frac{5}{6} \times \frac{5}{6} = \frac{25}{36} \]

\[ \therefore \text{the probability that once at least the ace will be thrown is} \]

\[ 1 - \frac{25}{36} = \frac{11}{36} \]

Again, omitting to consider the aces, the number of chances of failing to throw one ace or two is

\[ \frac{10 \cdot 9}{2} = \frac{45}{2} \]

and the whole number of chances is

\[ \frac{12 \cdot 11}{2} = 66 \]

\[ \therefore \text{the probability of not throwing one ace at least is} \]

\[ \frac{45}{66} = \frac{15}{22} \]

Consequently

\[ P' = 1 - \frac{15}{22} = \frac{7}{22} \]

is the probability of throwing at least an ace in one trial with two dice, and we have

\[ P : P' :: \frac{11}{36} : \frac{7}{22} \]

\[ :: 11^2 : 7 \times 18 \]

\[ :: 121 : 126 \]

794. If \( \frac{a}{a+b} \) be the probability of an event's happening
in one trial; then by Wood's Algebra, p. 270, or any other elementary Treatise on the subject, it is shown that the probability of its happening at least \( t \) times in \( n \) trials is,

\[
\left( a + b \right)^n = a^n + n a^{n-1} b + n \frac{n-1}{2} a^{n-2} b^2 + \text{&c. to } n-t+1 \text{ terms}
\]

Now by the problem, \( a = 1, b = p - 1 \)

\[
\therefore \quad P = \frac{1 + n(p-1) + n \frac{n-1}{2} (p-1)^2 + \text{&c. to } n-1+1 \text{ terms}}{p^n}
\]

795. To generalize the problem somewhat, suppose, other circumstances remaining the same, that the number of bowls each plays with is \( n \).

Then if the probability of \( A \) getting one ball or more at the first end is \( \frac{1}{2} \); of his getting a second in is \( \frac{n-1}{2n-1} \). Consequently that of his winning one precisely is

\[
\frac{1}{2} - \frac{1}{2} \frac{n-1}{2n-1} = \frac{n}{2(2n-1)}
\]

Hence the probability of \( A \)'s winning in the above manner is

\[
P = \frac{n}{4(2n-1)} \quad \ldots \quad (1)
\]

Again, since the probability of \( A \)'s winning two or more the first end is

\[
\frac{1}{2} \cdot \frac{n-1}{2n-1}
\]

and that of his winning three or more is

\[
\frac{1}{2} \cdot \frac{n-1}{2n-1} \cdot \frac{n-2}{2n-2} = \frac{n-2}{4(2n-1)}
\]

\[
\therefore \quad \text{the probability of his winning two exactly is}
\]

\[
\frac{1}{2} \cdot \frac{n-1}{2n-1} - \frac{n-2}{4(2n-1)} = \frac{n}{4(2n-1)}
\]
PROBABILITIES.

But since he then wants one of the game and B two, if B wins one at an end, they become equal; hence

B's chance of winning this way is

\[ \frac{1}{4} \cdot \frac{n}{2n-1} \]

and his chance of winning by two or more is

\[ \frac{1}{2} \cdot \frac{n-1}{2n-1} \]

\[ \therefore \text{B's chance of winning is} \]

\[ \frac{1}{4} \cdot \frac{n}{2n-1} + \frac{1}{2} \cdot \frac{n-1}{2n-1} = \frac{3n-2}{8n-4} \]

Hence A's chance of winning the game after winning two the first end is

\[ 1 - \frac{3n-2}{8n-4} = \frac{5n-2}{8n-4} \]

and his chance of winning this second way is

\[ P' = \frac{n}{4(2n-1)} \times \frac{5n-2}{8n-4} \]

\[ = \frac{n(5n-2)}{16(2n-1)^2} \] \hspace{1cm} (3)

Again, the probability of A's winning three or more at the first end is

\[ \frac{1}{2} \cdot \frac{n-1}{2n-1} \cdot \frac{n-2}{2n-2} = \frac{1}{4} \cdot \frac{n-2}{2n-1} \]

A may win this way and we have

\[ P'' = \frac{1}{4} \cdot \frac{n-2}{2n-1} \] \hspace{1cm} (3)

Again, suppose B to win one bowl *precisely* at the first end, then A wants three and B one of the game. Now to calculate A's chance on this supposition, we have

A's chance of winning precisely at first end

\[ \frac{1}{2} \cdot \frac{n}{2n-1} \]
and his chance of winning the game afterwards

\[ \frac{3n-2}{8n-4} \]

\[ \frac{n \cdot (3n-2)}{8(2n-1)^2} \]

is part of A's chance of winning on this hypothesis. But A's chance of getting two precisely, the first end, being

\[ \frac{1}{4} \cdot \frac{n}{2n-1} \]

which makes him even with B gives

\[ \frac{1}{8} \cdot \frac{n}{2n-1} \]

for the second part of A's chance.

Also A's chance of winning three or more, viz.

\[ \frac{1}{2} \cdot \frac{n-2}{2n-1} \]

\[ \frac{1}{4} \cdot \frac{1}{2n-1} \]

\[ \frac{n-2}{2n-1} \]

\[ \frac{1}{2} \cdot \frac{n-2}{2n-1} \]

gives the other part of his chance of winning the game. Hence the probability of A winning the game this third way is

\[ P' = \frac{n}{2} \cdot \left( \frac{n \cdot (3n-2)}{8 (2n-1)^2} + \frac{1}{8} \cdot \frac{n}{2n-1} + \frac{1}{4} \cdot \frac{n-2}{2n-1} \right) \]

\[ = \frac{n (9n^2-13n+4)}{16 (2n-1)^3} \]

Hence the total value of A's chance of winning is

\[ P + P' + P'' = \frac{n}{4 (2n-1)} + \frac{n \cdot (5n-2)}{16 (2n-1)^2} + \frac{1}{4} \cdot \frac{n-2}{2n-1} \]

\[ + \frac{n \cdot (9n^2-13n+4)}{16 (2n-1)^3} \]

\[ = \frac{n-1}{2 (2n-1)} + \frac{n (19n^2-22n+6)}{16 (2n-1)^3} \]

\[ = \frac{51n^3-86n^2+46n-8}{16 (2n-1)^3} \]

796. There are four ways in which

\[ \{1, 2, 3, 4, 5, 6\} \]

\[ \{1, 2, 3, 4, 5, 6\} \]

may be combined, taking one from each row, so as to make 5, viz.
1 + 4 = 5, 2 + 3 = 5, 3 + 2 = 5, 4 + 1 = 5; and in like manner it may be shewn that there are six ways in which seven may be formed from them.

Again, since each figure of the one line may be combined with each one of the other, there are on the whole

6 × 6, or 36 combinations.

Hence 36 − (6 + 4) = 26, 36 − 6 = 30, 36 − 4 = 32, are the chances for throwing neither 5 nor 7, for throwing the 5 and for throwing the 7 the first throw, respectively.

Hence in three throws the varieties are as follows,

The total = 36

The number of chances of throwing neither 5 nor 7 = 26;  
That of not throwing 7 once = 30;  
That of not throwing 5 once is 32.

Hence

36 − 30 − 32 = the number of chances for throwing 7 at least once, not excluding the chances for 5.  
32 − 26 = ditto for throwing 7 at least once without the possibility of throwing a 5.

∴ 36 − 30 − (32 − 26) = 4464 is the number of chances of throwing at least once in three trials.

∴ \[
\frac{4464}{36^3} = \frac{4464}{46656} = \frac{1116}{11664} = \frac{279}{2916} = \frac{31}{324}
\]

∴ the odds are 293 to 31.

797. That two of them, as A, B, precisely, will die in the time specified is

\[
\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}
\]

and four things taken two and two, may be combined six ways.

∴ the probability that some two of them, and no more will die, is

\[
\frac{6}{16} = \frac{3}{8}
\]
PROBABILITIES.

619

Again, that three will die precisely is found to be (in same manner)

\[ 4 \times \frac{1}{16} = \frac{1}{4} \]

and that all four will die is

\[ \frac{1}{16} \]

\[ \frac{6}{16} + \frac{4}{16} + \frac{1}{16} = \frac{11}{16} \]

gives the probability that two at least will die in the year.

798. The probabilities of A losing the first, the two first, &c., and the \( n \) first games are

\[ \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \text{ &c.,} \]

respectively; consequently the value of B's expectation is

\[ \frac{S}{2} + \frac{2S}{2^2} + \frac{4S}{2^3} + \text{ &c.} \]

\[ \frac{2^{n-1}S}{2^n} = \frac{nS}{2} \]

which is the equivalent A ought to receive.

799. The probabilities of their being alive are respectively

\[ 1 - \frac{5}{6} = \frac{1}{6} \quad \text{and} \quad 1 - \frac{4}{5} = \frac{1}{5} \]

\[ \frac{1}{6} \times \frac{1}{5} = \frac{1}{30} \]

is the probability of both being alive.

\[ 1 - \frac{1}{30} = \frac{29}{30} \]

is the probability that both will not be alive.

Again, the probability that both will be dead is

\[ \frac{5}{6} \times \frac{4}{5} = \frac{20}{30} = \frac{2}{3} \]
PROBABILITIES.

\[ 1 - \frac{2}{3} = \frac{1}{3} \]

is the probability that both will not be dead in the given time.

800. The chance of throwing the ace three specified times and of missing the other two is

\[ \frac{1}{6^3} \times \frac{5^2}{6^2} \]

and five things combined 3 and 3 give

5.4.3
2.3

\[ P = \frac{250}{6^5} \]

\[ \text{chance of throwing the ace three times precisely (without regard to order) in five trials.} \]

Similarly

\[ \frac{25}{6^5} \text{ and } \frac{1}{6^5} \]

are the chances of throwing the ace four times and five times precisely.

\[ P' = \frac{25}{6^5} \]

\[ \text{is, &c., and } P : P' :: 250 : 256 :: 125 : 128. \]

801. Taking one figure from each of these rows, and adding them

\[ 1, 2, 3, 4, 5, 6 \]

\[ 1, 2, 3, 4, 5, 6 \]

\[ 1, 2, 3, 4, 5, 6 \]

it will be found that there are 27 ways of making up the 10, and but 6 which give the number 5. Also 6^3 is the whole number of combinations.

\[ \therefore \text{the probabilities of throwing } 10 \text{ and } 5 \text{ with three dice are as} \]

\[ \frac{27}{6^3} : \frac{6}{6^3} :: 9 : 2. \]
802. Let the skill of the parties $A, B$ be as $a, b$; then
\[
\frac{a}{a+b} \quad \frac{b}{a+b}
\]
will represent the probabilities of either winning any one game.
Now $A$ may win in the following 10 ways, viz. by getting the
1 and 2, 1 and 3, 1 and 4, 1 and 5 games
2 and 3, 2 and 4, 2 and 5
3 and 4, 3 and 5
4 and 5
But
\[
P_{1,2} = \left(\frac{a}{a+b}\right)^2
\]
\[
P_{1,3} = \left(\frac{a}{a+b}\right) \times \frac{b}{a+b} \times \frac{a}{a+b} = \frac{a^2b}{(a+b)^3}
\]
\[
P_{1,4} = \frac{a^2}{(a+b)^2} \times \frac{b^2}{(a+b)^3} = \frac{a^2b^2}{(a+b)^5}
\]
\[
P_{1,5} = \frac{a^2}{(a+b)^2} \times \frac{b^3}{(a+b)^3} = \frac{a^2b^3}{(a+b)^5}
\]
\[
P_{2,3} = \frac{a^2}{(a+b)^2} \times \frac{b}{a+b} \quad P_{2,4} = \frac{a^2}{(a+b)^2} \times \frac{b^2}{(a+b)^3} \quad P_{2,5} = \frac{a^2b^3}{(a+b)^5}
\]
\[
P_{3,4} = \frac{a^2}{(a+b)^2} \times \frac{b^3}{(a+b)^3} \quad P_{3,5} = \frac{a^2}{(a+b)^2} \times \frac{b^3}{(a+b)^3}
\]
\[
P_{4,5} = \frac{a^2b^3}{(a+b)^5}
\]
\[
\therefore P = \left(\frac{a}{a+b}\right)^2 \times \left\{1 + 2 \frac{b}{a+b} + 3 \frac{b^2}{(a+b)^2} + 4 \frac{b^3}{(a+b)^3}\right\}
\]
In the problem, $a = 2, b = 3$.
\[
\therefore P = \frac{4}{25} \times \left\{1 + \frac{6}{5} + \frac{27}{25} + \frac{108}{125}\right\}
\]
\[
= \frac{4}{25} \times \frac{518}{125} = \frac{2072}{3125}
\]
\[
\therefore P' = 1 - \frac{2072}{3125} = \frac{1053}{3125}
\]
the chances required.
803. That two of them specified will die, and the other
two be alive is \( \frac{1}{10 \times 10} \times \frac{9}{10} \times \frac{9}{10} = \frac{81}{10000} \) and the combi-
nations in 4 things taken, two and two is \( \frac{4 \times 3}{2} = 6 \).

\[ \therefore P = \frac{6 \times 81}{10000} \]
is the probability that some two and no more of them will die in
the time.

Again, that some three and no more will die is

\[ \frac{4 \times 9}{10000} \]

and that all four will die is

\[ \frac{1}{10000} \]

\[ \therefore P' = \frac{1 + 36 + 486}{10000} = \frac{523}{10000} \]
is the probability that some two at least will die.

\[ \therefore P : P' :: \frac{6 \times 81}{10000} : \frac{523}{10000} :: 486 : 523. \]

804. The probability of throwing an ace twice only in
three throws in a specified order is

\[ \frac{1}{36} \times \frac{5}{6} = \frac{5}{36 \times 6} \]

and there are \( \frac{3 \times 2}{2} = 3 \) different ways of being successful, \( \therefore \)

\[ \frac{5 \times 3}{6 \times 36} = \frac{5}{72} \]

the probability required.

805. The chance of his throwing it the first time is

\[ \frac{2}{36} = \frac{1}{18} \]
PROBABILITIES.

\[ \frac{17}{18} \] chance of not throwing it

and \[ \frac{17}{18^2} \] ditto of not throwing it once in four trials.

\[ \frac{18^4 - 17^4}{18^2} = \frac{21455}{104976} \] is the probability required.

806. Let \( x + y \) be the value of A's chance at first, \( x \) being that after having lost the first game, and first suppose \( p=2, q=1 \).

When A has lost the first game

\[ a \cdot \frac{x+y}{a+b} \] is his prospect of winning

and \( \therefore \ a \cdot \frac{x+y}{a+b} = x \), \( \therefore \ y = \frac{bx}{a} \).

Again, before they begin, A has \( a \) chances for 3 counters, and \( b \) chances for \( x \) counters.

\[ \therefore \ 3a + bx \] \[ \frac{a+b}{a+b} = x + y = x + \frac{bx}{a} \]

\[ \therefore \ x = \frac{3a^3}{a^2 + ab + b^2} \]

and \( x + y = 3. \frac{a^2 + ab}{a^2 + ab + b^2} \)

and \( \therefore \ B's \) expectation is

\[ 3 - x + y = 3. \frac{b^2}{a^2 + ab + b^2} \]

Again, let \( p = 3, q = 1 \).

Then putting \( x + y + z = A's \) expectation at first, \( x + y \) after having lost the first game, and \( x \) after having lost the two first; by like reasoning we get

\[ a \cdot \frac{x+y}{a+b} = x \] and \( y = \frac{bx}{a} \)

\[ \frac{ax + ay + bx}{a+b} = x + y \] and \( z = \frac{by}{a} = \frac{b^2}{a^2} \)

\[ \therefore \ x + y + z = x + \frac{bx}{a} + \frac{b^2}{a^2} \).
PROBABILITIES.

Now it is evident that $A$ has at first $a$ chances for 4 counters, and $b$ chances for $x + y$ counters,

$$\therefore \frac{4a + b}{a + b} \cdot x + y = x + y + z$$

which gives

$$x = \frac{4a^3}{a^3 + ba^2 + b^2a + b^3}$$

$$\therefore x + y + z = 4 \cdot \frac{a^3 + a^2b + b^2a}{a^3 + ba^2 + b^2a + b^3}$$

and $B$'s expectation is

$$A. \quad \frac{b^3}{a^3 + ba^2 + b^2a + b^3}$$

Hence if $p = p$ and $q = 1$, we get

$$P = \frac{a^n + a^{n-1}b + \ldots + ab^n}{a^n + a^{n-1} + \ldots + b^n} \times (n + 1) = \frac{n + 1}{a^{n+1} - b^{n+1}}$$

$$Q = \frac{b^na - b^{n+1}}{a^{n+1} - b^{n+1}} \times (n + 1).$$

Again, let $p = p, q = q$.

Then $x$ being the value of $A$'s expectation when he has lost all but one counter, we have

$$x + \frac{bx}{a} + \frac{b^2x}{a^2} + \ldots + \frac{b^{n-1}x}{a^{n-1}} = \frac{x \left( \frac{b}{a} \right)^n - x}{\frac{b}{a} - 1}$$

the value of $A$'s expectation at first, and

$$x + \frac{bx}{a} + \frac{b^2x}{a^2} + \ldots + \frac{b^{n+1}x}{a^{n+1}} = \frac{x \left( \frac{b}{a} \right)^{n+1} - x}{\frac{b}{a} - 1}$$

for $A$'s expectation when $B$ has but one left.

Hence $A$ has $a$ chances for the whole $p + q$, and $b$ chances for the above expectation, $\therefore$

$$\frac{(p + q)a^{n+1} \times (b - a)}{b^{n+1} - a^{n+1}}$$

and $A$'s original expectation becomes
\[ P = \frac{a^n \times (b^n - a^n)}{b^{n+1} - a^{n+1}} \times (p + q) \]

the probability required.

and \( \therefore \) B's is

\[ Q = \frac{b^{n+1} - a^{n+1}}{b^{n+1} - a^{n+1}} \times (p + q) \]

\[ \therefore \frac{P}{Q} = \frac{a^n (b^n - a^n)}{(b^n - a^n) b^n} \]

\[ \therefore \frac{b^n - a^n}{a^n} = \frac{b^n}{a^n} \]

Make \( b = a + x \)

Then \( \frac{P}{Q} = \frac{q a^{x-1} + \ldots}{p b^{x-1} + \&c.} \)

\[ \therefore \frac{q a^{x-1} + x R}{p a^{x-1} + x R} = \frac{b^n}{a^n} \]

Let \( x = 0, \) or \( a = b, \)

and we get

\[ \frac{P}{Q} = \frac{p}{q} \]

a very remarkable result, and which shews most clearly that a person always playing at games of chance for the same stake, or at games of skill with antagonists of equal skill with himself, must in the end, be utterly ruined.

At the rouge et noir tables, for instance, even were the play fair and strictly honourable, and the banker to have no advantage (which he always has), continual play for the same stakes, must leave the wealthiest person in the world without a sou. It may also be remarked, that the safest play, is constantly to stake the same sum and adhere to the same colour.

When \( b \) is very small in respect to \( a \)

\[ \frac{P}{Q} = 1 \left( \frac{b}{a} \right)^n \text{ nearly.} \]

807. Since \( p \) of them are taken at once, and there are but \( p \) of the balls specified, there is evidently but one way of being successful, and there are in all

\[ \text{VOL. I.} \]

28
ways. Consequently the probability required is
\[
P = \frac{n!}{(n-1)! \cdots (n-p+1)!} \times \frac{1}{2 \cdots p}
\]

The question may be generalized by stating it: "required the probability that in taking \( p \) at a time from a bag containing \( n \) balls, \( n' \) of which are marked or specified, \( p' \) of the \( p \) shall be of the sort specified."

Combining the \( n' \) marked balls \( p' \) and \( p' \), we have
\[
n'Cp'
\]
and also combining the \( n - n' \) plain balls \( p - p' \) and \( p - p' \) together we get
\[
(p - n') \times (p - p')
\]
for the number of ways in which they can form variations, and each of these lots being added to each of the marked allotments make up all the ways in which \( p' \) of the marked balls can appear; which are the number of cases favourable to the event; and the whole number of combinations is
\[
nCp
\]
The probability required is
\[
\frac{nCp'}{nCp} \times \frac{n - n' \times p - p'}{nCp}
\]
or
\[
P = \frac{n!}{(n-1)! \cdots (n-p+1)!} \times \frac{1}{2 \cdots p'}
\]

This formula is very extensively useful. It will determine the probability of your getting, for instance, \( p' \) of the \( n' \) prizes in the State Lottery consisting of \( n \) tickets, if you have purchased \( p \) tickets therein. It will, moreover, give the chances of having so many certain cards trumps, &c. &c. at whist or other games.
808. By taking one figure from each of the lines, and adding them

\[
1.2.3.4.5.6
\]

the number of ways in which we can make 7 is 6, and the whole number of combinations is 6^*.

\[
\therefore \, P = \frac{6}{6^*} = \frac{1}{6}
\]

the probability of throwing 7 in any one trial. Hence the chance of throwing 7 twice precisely out of three times together in a given order is \(\frac{1}{6^*} \times \frac{5}{6}\); and there are \(\frac{3 \times 2}{2} = 3\) different orders.

\[
\therefore \, P' = \frac{5}{3} \cdot \frac{1}{6^*}
\]

is the chance of throwing 7 twice exactly in three trials.

\[
\therefore \, P : P' :: \frac{1}{6} : \frac{5}{3} :: \frac{1}{6^*} : 18 : 5.
\]

809. That A, B happen it is

\[
\frac{p}{p+q}, \quad \frac{r}{r+s}
\]

that they fail will be

\[
\frac{q}{p+q}, \quad \frac{s}{r+s}
\]

respectively.

Hence that A happens and B fails

\[
is \frac{p}{p+q} \times \frac{s}{r+s}
\]

and then that B happens and A fails is

\[
\frac{r}{r+s} \times \frac{p}{p+q}
\]

and then that A happens and B fails is

\[
\frac{p}{p+q} \times \frac{s}{r+s}
\]

&c., &c., &c.
PROBABILITIES.

that in \(2n\) trials A and B happen alternately is

\[
\frac{p^n q^n}{(p + q)^{2n}} \times \frac{q^n}{(r + s)^{2n}}
\]

the probability required.

810. The chance of throwing an ace any three specified throws and missing it the other two is

\[
\frac{1}{6^3} \times \frac{5^2}{6^2}
\]

and there are \(5 \times 4 \times 3\) ways of doing this.

\[
\therefore \frac{10 \times 5^2}{6^5} = \frac{2500}{7776} = \frac{1250}{3888} = \frac{625}{1944}
\]

the probability required.

811. The probabilities of the events, happening and failing are respectively

\[
\frac{a}{a + b}, \quad \frac{b}{a + b}
\]

\[
\therefore \frac{a^p}{(a + b)^p} \times \frac{b^q}{(a + b)^q} = \text{the probability of its happening } p
\]

specified trials and failing in the other \(q\) trials; but this can take place

\[
(p + q) \frac{C_p}{(a + b)^p}
\]

different ways

\[
\therefore P = \frac{a + q}{a + q - 1} \ldots \frac{q + 1}{1 \ldots p} \times \frac{a^p b^q}{(a + b)^{p + q}}
\]

Again to find the number of trials necessary to make it even whether the event happens or fails.

let \(x\) be the number required.

then \(\frac{a^x}{(a + b)^x} = \frac{1}{2}\)

and \(x = \frac{12}{la - (a + b)} = \frac{12}{la + b - la}\)
PROBABILITIES. 629

812. The number of ways of drawing 1, 3, 5, 7, &c. are

\[ N = n + \frac{n \cdot n - 1}{2} \cdot \frac{n - 3}{3} + \&c. 1 \text{ (to } \frac{n}{2} \text{ or } \frac{n + 1}{2} \text{ terms)} \]

those of drawing 2, 4, 6, 8 &c. are

\[ N' = \frac{n \cdot n - 1}{2} + \frac{n \cdot n - 2 \cdot n - 3 \cdot n - 4}{2 \cdot 3 \cdot 4} + \&c. 1 \text{ (to } \frac{n}{2} \text{ or } \frac{n - 1}{2} \text{ terms).} \]

Now \[ N = \left(\frac{1 + 1}{2}\right)^n + \left(\frac{1 - 1}{2}\right)^n = 2^{n-1} \]

and \[ N' = \left(\frac{1 + 1}{2}\right)^n - \left(\frac{1 - 1}{2}\right)^n = 2^{n-1} - 1 \]

\[ N + N' = 2^n - 1 \]

\[ \frac{N}{N + N'} = \frac{2^{n-1}}{2^n - 1} \quad \frac{N'}{N + N'} = \frac{2^{n-1} - 1}{2^n - 1} \]

and \[ N : N' :: 2^{n-1} : 2^{n-1} - 1. \]

Hence it appears more probable, that an odd number of balls should be taken from the bag than an even number, by the quantity \[ \frac{1}{2^n - 1}. \]

813. The probabilities of A, B, C, D getting a knave the first card are

\[ \frac{4}{52}, \frac{4}{51}, \frac{4}{50}, \frac{4}{49} \]

supposing B, C, D to have the opportunity. \( A_1 = \frac{1}{13} \) is the first part of A's chance (1). But if this does not succeed, A's second chance will depend upon A, B, C, D each failing; the probability of which is

\[ (1 - \frac{4}{52}) \cdot (1 - \frac{4}{51}) \cdot (1 - \frac{4}{50}) \cdot (1 - \frac{4}{49}) = \frac{48.47.46.45}{52.51.50.49} = \frac{4.9.28.47}{5.13.17.49} \]

where \( B_1 \)

whence A's second chance is

\[ A_2 = \frac{4}{48}, \quad B_1 = \frac{1}{12}, \quad B_1 = \frac{3.23.47}{5.13.17.49} \]
Again the probability that two rounds will be dealt without turning a knave is

\[ B_1 \times \frac{44.43.42.41}{48.47.46.45} = \frac{11.43.41}{5.5.13.17.7} \ldots B_2 \]

\[ A_3 = \frac{4}{44}, B_2 = \frac{43.41}{7.5.5.13.17} \]

Again, that three rounds fail is

\[ B_2 \times \frac{40.39.38.37}{44.43.42.41} = \frac{38.37}{5.7.7.17} \ldots B_3 \]

\[ A_4 = \frac{4}{40}, B_3 = \frac{1}{10}, B_3 = \frac{19.37}{5.5.7.17} \]

Proceeding in like manner with the other rounds, we get

\[ B_4 = \frac{9.11}{5.7.13}, A_5 = \frac{4}{35} = \frac{11}{5.7.13} \]

\[ B_5 = \frac{8.29.31}{7.17.13.35}, A_6 = \frac{4}{32} = \frac{29.31}{7.13.17.35} \]

\[ B_6 = \frac{9}{7.17.25}, A_7 = \frac{4}{28} = \frac{9}{7.17} \]

\[ B_7 = \frac{6.11.23}{7.13.17.25}, A_8 = \frac{4}{24} = \frac{11.23}{7.13.17.25} \]

\[ B_8 = \frac{3.19}{5.7.13.7}, A_9 = \frac{4}{20} = \frac{3.19}{5.5.7.7.13} \]

\[ B_9 = \frac{4}{5.17.7}, A_{10} = \frac{4}{16} = \frac{1}{5.7.17} \]

\[ B_{10} = \frac{9.11}{5.7.7.13.17}, A_{11} = \frac{4}{12} = \frac{3.11}{5.7.7.13.17} \]

\[ B_{11} = \frac{2}{5.7.17.13}, A_{12} = \frac{4}{5} = \frac{1}{5.7.13.17} \]

\[ B_{12} = \frac{1}{5.5.13.17.49}, A_{13} = \frac{1}{5.5.13.17.49} \]

Hence

\[ A_1 + A_2 + \ldots A_{13} = \frac{12757}{54145} \]
and A's expectation is worth
\[ \frac{12757}{54145} \text{£}, \text{or about 4s. 8¼d.} \]

814. The cards being arranged numerically thus
\[
\begin{align*}
2, 3, 4, 5, 6, 7, 8, 9, 10, 10, 10, 10, 11 \\
2, 3, 4, 5, 6, 7, 8, 9, 10, 10, 10, 10, 11 \\
2, 3, 4, 5, 6, 7, 8, 9, 10, 10, 10, 10, 11 \\
\end{align*}
\]

it is easily found that
\[ 29 = (11 + 10 + 8) \text{ in } 4 \times 4 \times 4 \times 4 \text{ different ways.} \]
\[ = (11 + 9 + 9) \text{ in } 4 \times \frac{4^3}{3} \text{ different ways.} \]
\[ = (10 + 10 + 9) \text{ in } 4 \times \frac{16}{3} \times \frac{15}{2} \text{ . . . . . . .} \]

: the different ways of making up 29 are in number
\[ 256 + 24 + 480 = 760. \]

Again
\[ 19 = 11 + 6 + 2 \text{ in } 4^3 \text{ different ways} \]
\[ = 11 + 5 + 3 \text{ in } 4^3 \]
\[ = 11 + 4 + 4 \text{ in } 4^3 \]
\[ = 10 + 7 + 2 \text{ in } 4^3 \]
\[ = 10 + 6 + 3 \text{ in } 4^3 \]
\[ = 10 + 5 + 4 \text{ in } 4^3 \]
\[ = 9 + 8 + 2 \text{ in } 4^3 \]
\[ = 9 + 7 + 3 \text{ in } 4^3 \]
\[ = 9 + 6 + 4 \text{ in } 4^3 \]
\[ = 9 + 5 + 5 \text{ in } 4^3 \]
\[ = 8 + 8 + 3 \text{ in } 4 \times \frac{4^3}{2} \]
\[ = 8 + 7 + 4 \text{ in } 4^3 \]
\[ = 8 + 6 + 5 \text{ in } 4^3 \]
\[ = 7 + 7 + 5 \text{ in } 4 \times \frac{4^3}{2} \]
\[ = 7 + 6 + 6 \text{ in } 4 \times \frac{4^3}{2} \]
PROBABILITIES.

\[ \therefore \text{the whole number of ways of producing 19 is} \]
\[ 4^3 \cdot 9 + 3^4 + 3 \cdot 4 \cdot 6 = 1160. \]

Again
\[ 9 = 5 + 2 + 2 \text{ in } 4 \cdot \frac{4 \cdot 3}{2} \text{ ways} \]
\[ = 4 + 3 + 2 \text{ in } 4^3 \text{ ways} \]
\[ \therefore \text{the whole number of ways in which 9 can be produced is} \]
\[ 24 + 64 = 88. \]

And the whole number of combinations in 52 things taken three and three, is
\[ \frac{52 \cdot 51 \cdot 50}{1 \cdot 2 \cdot 3} = 26 \cdot 17 \cdot 50 = 22200. \]

Hence the probabilities of drawing 29, 19, 9 respectively, are
\[ P = \frac{760}{22200} = \frac{38}{1110} = \frac{19}{555} \]
\[ P' = \frac{116}{2220} = \frac{58}{1110} = \frac{29}{555} \]
\[ P'' = \frac{88}{22200} = \frac{5550}{5550} = \frac{11}{2775} \]

And \[ P + P' + P'' = \frac{251}{2775} \]
is the probability of his getting 29, 19, or 9.
\[ \therefore 1 - \frac{251}{2775} = \frac{2524}{2775} \]
is the contrary, and \[ \therefore \text{we have} \]
\[ \frac{2524}{2775} : \frac{251}{2775} : 1 \text{ nearly against him.} \]

815. Since one bag contains more balls by \( m - n \) than the other, supposing we first draw from the former, the probability of hitting upon a ball contained in the latter, is
\[ \frac{n}{m} \]
and this happening, the probability required becomes that of drawing from the bag with \( n \) balls any specified one, or \( \frac{1}{n} \).
\[ \frac{n}{m} \times \frac{1}{n} = \frac{1}{m} \] is the probability required.

Otherwise.

There are evidently but \( n \) ways in which \( a, a; b, b; \ldots \) can come together, and the ways in which two sets of quantities containing \( m \) and \( n \) can be combined by taking one from each is

\[ \frac{m \times n}{mn} = \frac{1}{m} \]

the same as before.

816. Here \( P = \frac{(m-p')C(p-p') \times m'Cp'}{mCp} \) (see 806)

and \( m = 11, m' = 5, p = 7, p' = 3 \)

\[ \therefore P = \frac{\binom{6}{4} \times \binom{5}{3}}{\binom{11}{7}} = \frac{6 	imes 5 \times 4 \times 3 \times 1 \times 2}{11 	imes 10 \times 9 \times 8} = \frac{5}{11} \]

the probability required.

817. The probability of throwing the six faces in any given order, is \( \frac{1}{6^6} \)

and there are \( 6 \times 5 \times 4 \times 3 \times 2 \times 1 \) ways in which the order can be varied; \( \therefore \) the probability is

\[ \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{6^6} = \frac{20}{6^4} = \frac{5}{2^4 3^4} = \frac{5}{4 \cdot 81} = \frac{5}{324} \]

818. Since \( A's \) skill is double of \( B's, \) his chance of winning is \( \frac{2}{3} \)

and the value of his expectation, supposing the stake a guinea, is \( \frac{2}{3} \times 2 \) guineas.
PROBABILITIES.

Now since C and A play with equal skill, if \( x \) denote C's stake and A's as before, the probability of A's winning \( 1 + x \) guineas is

\[
\frac{1}{2}
\]

and the value of his expectation is

\[
\frac{1}{2} (1 + x)
\]

and by the question we get

\[
\frac{4}{3} = \frac{1}{2} (1 + x)
\]

\[
\therefore x = \frac{8}{3} - 1 = \frac{5}{3}
\]

of a guinea.

819. Let \( \frac{x}{a+x} \) denote E's chance of winning the first game, then the probability of D's winning the first \( n \) games is

\[
\left( \frac{a}{a+x} \right)^n = \frac{1}{2}
\]

by the question.

\[
\therefore a^{2^n} = a + x
\]

\[
\therefore x = a \left( 2^n - 1 \right)
\]

and \( \frac{x}{a+x} = \frac{a(2^n - 1)}{a + a(2^n - 1)} = \frac{2^n - 1}{2^n} \)

the probability required.

820. Let generally \( pCq \) denote the number of combinations of \( p \) things taken \( q \) and \( q \) together; then the number of ways in which A may be thrown with \( m \) dice, is

\[
N = (A - 1). C. (m - 1)
\]

- \( m. C. 1 \times (A - 7). C. (m - 1) \)
+ \( m. C. 2 \times (A - 13). C. (m - 1) \)
- \( m. C. 3 \times (A - 19). C. (m - 1) \)
+ &c.

see Parisot's Calcul Conjectural, p. 55; and therefore the number of ways in which B may be thrown is
\[ N' = (B - 1) \times C. (m - 1) \]
\[ - m \times C. 1 \times (B - 7) \times C. (m - 1) \]
\[ + m \times C. 2 \times (B - 13) \times C. (m - 1) \]
\[ - m \times C. 3 \times (B - 19) \times C. (m - 1) \]
\[ + \&c. \]

Also it may easily be shewn that \( 6^n \) is the total number of different throws of the \( m \) dice. Consequently the probability of throwing \( A \) the first time is

\[ \frac{N}{6^n} \]

and that of not throwing it once in \( n \) throws is

\[ (1 - \frac{N}{6^n})^n \]

and \( \therefore \) the probability of throwing \( A \) once at least in \( n \) throws is

\[ P = 1 - (1 - \frac{N}{6^n})^n \] \( \ldots \ldots \ldots (a) \)

In the same way it is shewn that the chance of throwing \( B \) at least once in \( n \) throws is

\[ P' = 1 - (1 - \frac{N'}{6^m})^n \] \( \ldots \ldots \ldots (b) \)

Hence the probability of throwing both \( A \) and \( B \) with \( m \) dice in \( n \) throws, at least once, is

\[ P + P' = 2 - (1 - \frac{N}{6^n})^n - (1 - \frac{N'}{6^m})^n \]

By way of example let it be required to find the chance of throwing 20 and 16 at least once with 4 dice in 6 throws.

Here
\[ N = 19, C. 3 \]
\[- 4, C. 1 \times 13, C. 3 \]
\[ + 4, C. 2 \times 7, C. 3 \]
\[ = \frac{19.18.17}{2.3} - 4 \times \frac{13.12.11}{2.3} + \frac{4.3}{2} \times \frac{7.6.5}{2.3} = 35. \]

\[ N' = \frac{15.14.13}{2.3} - 4 \times \frac{9.8.7}{2.3} + \frac{4.3}{2} \times \frac{3.3.1}{2.3} = 125. \]

\[ \therefore P = 1 - \left(1 - \frac{35}{6^2}\right)^6 = \frac{6^{24} - (6^4 - 35)^6}{6^{84}} \]
and \( P' = 1 - \left(1 - \frac{125}{6^2}\right)^6 = \frac{6^{24} - (6^4 - 125)^6}{6^{24}} \)
and so on.

821. Let the stake, each game, be \( S \); then the probability of B's winning any specified game being
\[
\frac{m}{m + 1}
\]
his expectation is worth
\[
\left(\frac{2m}{m + 1} - 1\right) S = \frac{m - 1}{m + 1} S;
\]
and the probability of his having to play the \( n \)th game being
\[
\left(\frac{m}{m + 1}\right)^{n-1}
\]
his gain on that game is worth
\[
\left(\frac{m}{m + 1}\right)^{n-1} \times \frac{m}{m + 1} \times S.
\]
Hence the whole value of his expectation is
\[
\frac{m - 1}{m + 1} S \times \left\{ 1 + \frac{m}{m + 1} + \left(\frac{m}{m + 1}\right)^2 + &c. \ infty \right\}
\]
which is
\[
\frac{m - 1}{m + 1} S \times (m + 1) = (m - 1) S.
\]

822. The whole number of combinations is
\[
\frac{8 \times 7 \times 6}{2 \times 3} = 56,
\]
out of which we have
1 consisting of 20\(£\) and two 5\(£\).
\[
\frac{5 \times 4}{2} = 10 \text{ of a 20} \(£\) and two \(1 \£\).
\]
5 of two fives and one \(1 \£\).
\[
2 \times \frac{5 \times 4}{2} = 20, \text{ of one 5} \(£\) and two \(1 \£\).
\]
\[
\frac{5 \times 4 \times 3}{2 \times 3} = 10 \text{ of three} \(1 \£\).
\]
and 10 of a 20\(£\), a 5\(£\), and a \(1 \£\).
Hence his expectations of winning 30 £, 11 £, 3 £, 26 £, are
\[
\begin{align*}
\frac{1}{56}, & \quad \frac{10}{56}, \quad \frac{5}{56}, \quad \frac{20}{56}, \quad \frac{10}{56}, \quad \frac{10}{56},
\end{align*}
\]
respectively; and the value of his expectation is therefore
\[
\frac{30 + 220 + 55 + 140 + 30 + 260}{56} £ = \frac{735}{56} £ = \frac{105}{8} £ = £13\ 2s.\ 6d.
\]
The sides of a right angle is \( x^2 + y^2 = r^2 \).

Eq. of rod, origin at \( t \), is \( y = ax + b \).

Since for \( y = 0 \), \( x = -\frac{b}{a} \), and for \( x = 0 \), \( y = b \), \( b^2 + \frac{b^2}{a^2} = r^2 \). With this eliminate \( a \).

\[ y = \left(\frac{b^2}{a^2} - b^2\right)^{\frac{1}{2}} + b. \] Differentiate

\[ y = \frac{1}{2} \left(\frac{b^2}{a^2} - b^2\right)^{\frac{1}{2}} \frac{1}{a^2} . \]

**MISCELLANIES.**

823. The line which passes through \( O \) (Fig. to enunciation) resting against the wall and ground at equal distances from the bottom of the wall \( B \), is the length of the largest ladder required; for if the longest were longer than this ladder, it would fall beyond \( O \) when in that position. Hence having the co-ordinates \( a, b \), of the point \( O \) referred to \( A \), we get for the parts of the ladder on each side of \( O \) the lengths

\[ a \sqrt{2}, b \sqrt{2} \]

and the length required is therefore

\[ (a + b) \sqrt{2} \]

This solution is wrong.

See Alexander's Calculus, \( p. 50 \).

824. Let \( r \) be the common ratio of the geometrical series; then the successive values are

\[ x, xr, xr^2, xr^3, \cdots, xr^n, \cdots \]

and the general increment of the ratio is

\[ \frac{xr^{n+1} - xr^n}{xr^n} = r - 1 \]

a constant quantity.

825. The coefficient of \((n + 1)^{th}\) term of \((a + x)^n\) is

\[ \frac{2n \cdot (2n - 1) \cdots \frac{n+1}{2}}{2 \cdot 3 \cdots n} \]

But

\[ (a + x)^n = a^n + na^{n-1}x + \frac{n-1}{2} a^{n-2} x^2 + \&c. \]

\[ (a + x)^n = a^n + na^{n-1}x + \frac{n-1}{2} a^{n-2} x^2 + \&c. \]

and multiplying them together the coefficient of \( a^n x^n \) will be found to be
on the supposition that $b$ only is variable.

\[ : \quad b = n \left(1 - \left(\frac{x}{2}\right)^2\right)^{\frac{1}{2}} \]

Substituting gives: \[ \begin{align*}
\frac{x^3}{z^2} + y^3 &= \frac{n^2}{3} \\
\text{in (3)} &\end{align*} \]

\[ \text{MISCELLANIES.} \]

\[ 1 + n^2 + \left(n \cdot \frac{n-1}{2}\right)^2 + \text{&c. to } n+1 \text{ terms,} \]

and the middle term of \((a + x)^n\) being the \((n + 1)\)th, its coefficient is followed by \(a^n x^n\).

\[ \therefore 1 + n^2 + \left(n \cdot \frac{n-1}{2}\right)^2 + \text{&c.} = \text{that coefficient} \]

\[ = \frac{2n \cdot 2n - 1 \ldots \ldots n+1}{1 \cdot 2 \cdot 3 \ldots \ldots n} \]

\[ = \frac{2n \cdot 2n - 1 \ldots \ldots n+1}{1 \cdot 2 \cdot 3 \ldots \ldots n} \times \frac{n \cdot (n-1) \ldots \ldots 2 \cdot 1}{n \cdot (n-1) \ldots \ldots 2 \cdot 1} \]

\[ = \frac{2n \cdot (2n - 2) \ldots \ldots 4 \cdot 2 \times 1 \cdot 3 \cdot 5 \ldots \ldots 2n - 1}{(1 \cdot 2 \cdot 3 \ldots \ldots n)^2} \]

\[ = 2^n \times \frac{1 \cdot 3 \cdot 5 \ldots \ldots 2n - 1}{1 \cdot 2 \cdot 3 \ldots \ldots n} \]

Q. E. D.

826. Let \(a^m b^n = c\). Then

\[ mxa + nzlb = lc \]

and \(mla \times dx + nlb \times dz = 0\).

Also by the question,

\[ (mr + n) \times (nz + m) = \text{max.} \]

\[ \therefore mdx \cdot (nz + m) + nzdz \cdot (mx + n) = 0 \]

\[ \therefore \frac{m}{n} \frac{dx}{dz} = \frac{lb}{la} = \frac{mx + n}{nz + m} \]

\[ a^{m+n} = b^{m+n}. \]

Q. E. D.

Otherwise,

since \(mx = \frac{lc - lb \times nz}{la}\), we have

\[ \left( \frac{lc - lb \times nz}{la} + n \right) \left( nz + m \right) = \text{max.} \]

\[ \therefore \frac{lb}{la} \times n \times (nz + m) = \left( \frac{lc - lb \times nz}{la} + n \right) n \]

\[ \therefore n^2 \frac{lb}{la} z + nm \frac{lb}{la} = \frac{n^2}{la} \frac{lc}{la} - n^2 \frac{lb}{la} z + n^2. \]
Hence

\[ n + mx = (m + nz) \frac{lb}{la} \]

and \( \therefore a^{n + mz} = b^{n + zn} \).

827. Let \( 2 \cos z = x + \frac{1}{x} \). Then

\[
1 + n \cos z = 1 + \frac{n}{2} \left( x + \frac{1}{x} \right)
= \left( \frac{n}{2} x^2 + x + \frac{n}{2} \right) \times \frac{1}{x}
= \frac{n}{2x} \times \left( x^2 + \frac{2}{n} x + 1 \right)
= \frac{n}{2x} \times \left( x + 1 + \sqrt{1 - \frac{1}{n^2}} \cdot \left( x + 1 - \sqrt{1 - \frac{1}{n^2}} \right) (x + B) \right)
= \frac{nB}{2} \left( 1 + Bx \right) \left( 1 + \frac{B}{x} \right)
\]

\( \therefore l \left( 1 + n \cos z \right) = l \cdot \frac{nB}{2} + l \cdot (1 + Bx) + l \cdot \left( 1 + \frac{B}{x} \right) \)

\[= l \cdot \frac{nB}{2} + Bx - \frac{B^2 x^2}{2} + \frac{B^3 x^3}{3} \quad \text{&c.} \]
\[+ \frac{B}{x} - \frac{B^2}{2x^2} + \frac{B^3}{3x^3} \quad \text{&c.} \]
\[= l \cdot \frac{nB}{2} + B \left( x + \frac{1}{x} \right) - \frac{B^2}{2} \left( x^2 + \frac{1}{x^2} \right) \quad \text{&c.} \]
\[= l \cdot \frac{nB}{2} + 2B \cos z - \frac{2B^2}{2} \cos 2z + \frac{2B^3}{3} \cos 3z \quad \text{&c.} \]

828. Let \( u = \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \ldots \frac{1}{w} \) be the series of the reciprocals of prime numbers; then by Legendre, p. 397, we have

\[ u = \log \left( \log w - 0.08366 \right) - 0.2215. \]
Again, we have
\[ x + \frac{x^2}{2} + \frac{x^3}{3} + \&c. = - \log(1 - x) \]
and when \( x = 1 \),
\[ u' = 1 + \frac{1}{2} + \frac{1}{3} + \&c. = - \log 0 \]
\[ = \log \frac{1}{0} = \log \infty. \]

Also when \( w \) is indefinitely great or \( \infty \), we have
\[ u = \log (\log \infty) = \log u' \]
\[ \therefore u' = e^u = (2.718, \&c.)^\infty \]
and \( \frac{u'}{u} = \frac{(2.719, \&c.)^\infty}{\infty} \)
\[ = \frac{(1+A)^\infty}{1 + \infty \times A + \frac{\infty^2}{2} A^2 + \&c.} \]
\[ = A + \frac{\infty^2}{2} A^2 + \frac{\infty^3}{3} A^3 + \&c. \]
\[ = \infty \times \left( \frac{A^2}{2} + \frac{A^3}{3} + \&c. \right) \]

so that \( u' \) is indefinitely greater than \( u \). Q. E. D.

829. Let \( P \) (draw the figure) be the point within the circumference, \( aPb, bPc \), the equal angles subtended by the arcs \( ab, bc, ab \), being nearer to the diameter passing through \( P \) than \( bc \); then \( bc \) is \( > ab \).

For let \( cP, bP, bP \), be produced to meet the circumference in \( c', b' \), and join \( ab', bc' \); then since \( Pb' > Pc' \) and the angle \( bPc' = \) the angle \( b'Pa, \therefore \) the \( \angle b' \) is \( < \) the angle \( c' \), or since equal \( \angle \) are subtended by equal arcs, \( ab \) is \( < bc \).

830. Since \( x = \frac{e^a - e^b}{2a} \) and \( v = \pm \sqrt{-1} \)
\[
\therefore x = \frac{e^{i\sqrt{a^2 - x^2}} - e^{-i\sqrt{a^2 - x^2}}}{2i\sqrt{a^2 - x^2}} = \sin z
\]
\[
\therefore dz = \frac{dx}{\sqrt{1 - x^2}}.
\]

831. Make
\[
P = \sqrt{(2a^3 x - x^2)} - \sqrt{(ax^3)}
\]
and \[
Q = a - \sqrt{ax}
\]

Then
\[
\frac{dP}{dx} = \frac{a^3 - 2x^3}{\sqrt{(2a^3 x - x^2)}} - \frac{3}{2} \sqrt{\frac{ax}{x}}
\]
and
\[
\frac{dQ}{dx} = - \frac{1}{2} \sqrt{\frac{a}{x}}
\]
and these, when \(x = a\), become
\[
-3a^3 - \frac{3a^2}{2}, \text{ and } -\frac{1}{2}
\]
\[
\therefore \frac{dP}{dQ} = 3a(2a^2 + 1) \text{ when } x = a
\]
which is the value required.

Again,
\[
d \cdot \frac{\sqrt{a+x} + \sqrt{a-x}}{\sqrt{a+x} - \sqrt{a-x}} = d \cdot \frac{(\sqrt{a+x} + \sqrt{a+x})^2}{2x}
\]
\[
= 2d \cdot \frac{(\sqrt{a+x} + \sqrt{a-x}) - 2dx}{x}
\]
\[
= dx \left\{ \frac{1}{\sqrt{a+x}} - \frac{1}{\sqrt{a-x}} \right\} - \frac{2dx}{x}
\]
\[
= dx \left\{ \frac{\sqrt{a+x} - \sqrt{a+x}}{\sqrt{a^2 - x^2} \cdot \sqrt{a+x} + \sqrt{a-x}} \right\} - \frac{2}{x}
\]
\[
= dx \left\{ \frac{(\sqrt{a-x} - \sqrt{a+x})^2}{-2x\sqrt{a^2 - x^2}} - \frac{2}{x} \right\}
\]
which may be farther reduced.
832. By the nature of the series we have (calling the coefficients \( a_1, a_2, \ldots, a_n \), &c., and the scale of notation being \( f_1 + f_2 + \ldots, \))

\[
\begin{align*}
a_{n+1} &= f_1 a_n + f_2 a_{n-1} + \ldots, f_n a_1 \\
a_{n+2} &= f_1 a_{n+1} + f_2 a_n + \ldots, f_n a_2 \\
a_{n+3} &= f_1 a_{n+2} + f_2 a_{n+1} + \ldots, f_n a_3 \\
&\quad \&c. & \&c. \\
a_{2n} &= f_1 a_{2n-1} + f_2 a_{2n-2} + \ldots, f_n a_n \\
\therefore \Delta a_{n+1} &= f_1 \Delta a_n + f_2 \Delta a_{n-1} + \ldots, f_n \Delta a_1 \\
\Delta^2 a_{n+1} &= f_1 \Delta^2 a_n + f_2 \Delta^2 a_{n-1} + \ldots, f_n \Delta^2 a_1 \\
&\quad \&c. = & \&c.
\end{align*}
\]

and \( \Delta^n a_{n+1} = f_1 \Delta^n a_n + f_2 \Delta^n a_{n+1} + \ldots, f_n \Delta^n a_1 \).

Let \( \Delta^n a_n = 0 \); then

\[
f_1 + f_2 + \ldots, f_n = 1. \quad \text{Q. E. D.}
\]

833. \[
\begin{align*}
\int \frac{x^2}{\sqrt{a^2 - x^2}} \, dx &= \int \frac{x^2}{\sqrt{a^2}} \times (1 - \frac{x^2}{a^2})^{-\frac{1}{2}} \, dx \\
&= \frac{1}{a^2} \times \int \left\{ \frac{1}{3} x^3 + \frac{1}{2a^2} x^2 \frac{15}{2} \, dx + \frac{3}{4a^4} \frac{2^2}{2} \, dx + & \text{&c.} \right\} \\
&= \frac{1}{a^2} \times \left\{ \frac{2}{3} x^3 + \frac{1}{17a^4} x^{17} + \frac{3}{231a^{12}} x^{31} + & \text{&c.} \right\} + C.
\end{align*}
\]

which converges since \( a \) is \( > x \).

834. Let \( ABC \) (draw the fig.) be the \( \Delta \); with \( C \) as a centre and radius \( = \) \( CB \), \( CB \) being less than \( CA \) describe a circle cutting \( AC \) in \( E \), produce \( AC \) to \( D \), join \( BD, BE \), and draw \( EF \) parallel to \( BD \); then since

\[
\angle EBA = \angle CBA - \angle CEB = A - \frac{A + B}{2}
\]

\[
= \frac{B - A}{2}
\]

\( A \) and \( B \) denoting the angles at those points, we have

2 T 2
AD(=a+b) : AE(=a-b) :: DB : EF

:: EB . tan. \(\frac{A + B}{2}\) : EB . tan. \(\frac{A - B}{2}\)

:: tan. \(\frac{A + B}{2}\) : tan. \(\frac{A - B}{2}\)

835. Since \(\sqrt{\frac{(x + a)^3}{x - a}} = \text{minimum}\)

\(\therefore u = \frac{(x + a)^3}{x - a} = \text{min.}\)

\(\therefore \frac{du}{dx} = \frac{3(x + a)^2}{x - a} - \frac{(x + a)^3}{(x - a)^2} = 0\)

\(\therefore 3x - 3a = x + a\)

and \(x = 2a\).

Hence the minimum value is

\(\sqrt{\frac{27a^3}{a}} = 3a \sqrt{3}\).

836. \(d (\sin^{-1} 2y \sqrt{1 - y^2}) = d \left(\frac{2y \sqrt{1-y^2}}{\sqrt{1 - 4y^2(1-y^2)}}\right)\)

\[= \frac{2dy \left(\sqrt{1 - y^2} - \frac{y^2}{\sqrt{1 - y^2}}\right)}{\sqrt{1 - 4y^2 + 4y^4}} = 2dy \cdot \frac{1 - 2y^2}{\sqrt{(1 - 2y^2)^2}}\]

\[= 2dy.\]

837. Since \(13x + 14y = 200\), therefore

\(x + y + \frac{y}{13} = 15 + \frac{5}{13}\).

Let \(\frac{y - 5}{13} = w, w\) being any integer. Then

\(y = 13w + 5\)

\(\therefore x = 10 - 14w.\)
Hence if \( w \) be assumed = 0, 1, 2, 3, \&c., the values of \( y \) will be

5, 18, 31, \&c.

and the corresponding ones of \( x \) will be

10, - 4, - 18, \&c.

838. Let \( x^3 - x^2 - 8x + 12 = u \). Then by the question

\[
\frac{du}{dx} = 3x^2 - 2x - 8 = 0
\]

and \( x^2 - \frac{2}{3}x = \frac{8}{3} \)

\[
\therefore x = \frac{1}{3} \pm \sqrt{\frac{1}{9} + \frac{8 \times 8}{9}}
\]

\[
= \frac{1 \pm 5}{3} = 2 \text{ or } -\frac{4}{3}
\]

839.

\[
d\left( \frac{y}{\sqrt[3]{a^2} - y^2} \right) = dy \left\{ \frac{a^2 - y^2}{a^3 - y^3} \right\}
\]

\[
= \frac{y^2}{(a^3 - y^3)\sqrt[3]{a^2} - y^2} + \frac{3y^3 \sqrt[3]{a^2} - y^2}{(a^3 - y^3)^2}
\]

\[
= \frac{(a^2 - 2y^2)(a^3 - y^3) + 3a^2y^3 - 3y^3}{(a^3 - y^3)^2\sqrt[3]{a^2} - y^2}
\]

\[
= \frac{a^5 + 2a^2y^3 - 2a^3y^2 + y^5}{(a^3 - y^3)^2\sqrt[3]{a^2} - y^2}
\]

840. The several interests being \( \frac{r}{100}n, \frac{r}{100}(n - 1)2^3, \)

\( \frac{r}{100} \times (n - 2)3^3 \) \&c., \( \frac{r}{100}n^3 \) and the principals

1, 2, 3, \&c., \( n \)
the whole amount at the end of \( n \) years is

\[
S = 1 + \frac{r}{100} \cdot n + 2^3 \left( 1 + \frac{r}{100} \cdot \frac{n-1}{100} \right) + 3^3 \left( 1 + \frac{r}{100} \cdot \frac{n-2}{100} \right) + \ldots + n^3 \left( 1 + \frac{r}{100} \right)
\]

\[
&+ \frac{r}{100} \times \left( n + 2^3 \cdot \frac{n-1}{100} + 3^3 \cdot \frac{n-2}{100} + \ldots + n^3 \right)
\]

\[
= \left( 1 + 2^3 + \ldots + n^3 \right) \left( 1 + \frac{nr}{100} \right) - \left( 2^3 + 2 \cdot 3^3 + 3 \cdot 4^3 + \ldots + n^3 \right)
\]

\[
= (1 + \frac{nr}{100}) \sum n^3 - \sum n - \frac{n^3}{30} + C
\]

Let \( n = 1 \); then \( S = 1 + \frac{r}{100} \)

\[
\therefore C = 1 + \frac{r}{100} + \frac{1}{2}
\]

The farther reduction is left to the student.

841. Since \( u = \frac{x^3 (1-x^2)}{1-a^2x^2} = \text{max.} \)

\[
\frac{du}{dx} = \frac{2x(1-x^2)}{1-a^2x^2} - \frac{2x^3}{1-a^2x^2} + \frac{2a^2x^2(1-x^2)}{(1-a^2x^2)^2} = 0
\]

\[
\therefore (1-x^2)(1-a^2x^2) - x^3(1-a^2x^2) + a^2x^2(1-x^2) = 0
\]

\[
\therefore (1-2x^2)(1-a^2x^2) + a^2x^2(1-x^2) = 0
\]

Hence \( a^2x^4 - 2x^2 = -1 \)

and \( x^4 - \frac{2}{a^2} x^2 = -\frac{1}{a^2} \)

\[
\therefore x^2 = \frac{1 \pm \sqrt{(1-a^2)}}{a^2}
\]
and \( x = \pm \sqrt{1 \pm \sqrt{1 - a^2}} = \pm \frac{\sqrt{1 + a} \pm \sqrt{1 - a}}{a \sqrt{2}} \),

which gives the values required.

842. The 1000\(£\) stock at 110 = 1100\(£\) sterling = \(\frac{100}{84}\) 
\(\times\) 1100 = \(\frac{25}{21}\) \times 1100 in the Threes at 84. The interest of this 
at the end of six months is 
\(\frac{3}{200} \times \frac{25 \times 1100}{21} = \frac{33 \times 25}{42}\).

Now had the stock remained in the Fives, its worth (at 112) 
would have been 
1120 + \(\frac{1}{40}\) \times 1000 = 1120 + 25
\(= 1145\).

Hence \( x \) being the rate required, we have
\(\frac{x}{100} \times 1100 + \frac{33 \times 25}{42} = 1145\)
and \(\therefore x = -\frac{25}{14} + \frac{1145}{11}\)
\(= 16030 - 275\)
\(\frac{11 \times 14}{11 \times 14}\)
\(= 102 \frac{47}{54}\).

843. \(d \cdot l \cdot \frac{x}{\sqrt{1 + x^2}} = d \cdot lx - \frac{1}{2} d \cdot l (1 + x^2)\)
\(= \frac{dx}{x} - \frac{x \, dx}{1 + x^2} = \frac{dx}{x(1 + x^2)}\).

Also making
\[\sin \theta = 2x \sqrt{1 - x^2}\]
\[ \therefore d\theta \cos \theta = 2dx \sqrt{1 - x^2} - \frac{2x^2dx}{\sqrt{1 - x^2}} = \frac{2dx(1 - 2x^2)}{\sqrt{1 - x^2}}. \]

\[ \therefore d\theta = \frac{2dx(1 - 2x^2)}{\sqrt{1 - 4x^2 + 4x^4}} = 2dx. \]

844. Let \( u, u_1, u_2 \ldots u_n \) be the equidistant values of any function, then we know that (see Translation of Lacroix)

\[ \Delta^n u = u_n - \frac{n}{1} u_{n-1} + \frac{n(n-1)}{2} u_{n-2} - \&c. \pm nu_1, \mp u_0. \]

Hence, having all but one of these equidistant values, that one may be determined.

In the problem

\[ \begin{align*}
  u &= l \cdot 510 = 2.70757018 \\
  u_1 &= l \cdot 511 = 2.70842090 \\
  u_3 &= l \cdot 513 = 2.71011737 \\
  u_4 &= l \cdot 514 = 2.71096312 \\
\end{align*} \]

and \( u_x = \frac{4(u_1 + u_3) - (u + u_4)}{6} \)

\[ = 2.70926996. \]

845. Suppose two planes passing through the lines \( \perp \) to any third plane, and therefore parallel to one another. Hence a plane cuts two parallel planes at right \( \perp \) and the intersections (which are also the projections of the straight line in the question) are parallel.

846. Since \( a^x b^y c^z = \min. \)

\[ \therefore xla + ylb + zlc = \min. \]

and \( (x + 1) \cdot (y + 1) \cdot (z + 1) = Q \ldots (1) \]

Hence (see Vince, p. 20)

\[ \frac{dx}{dy} = - \frac{lb}{la} = - \frac{x+1}{y+1} \]
\[
\frac{dx}{dz} = -\frac{lc}{la} = -\frac{x+1}{z+1}
\]
\[
\frac{dy}{dz} = -\frac{lc}{lb} = -\frac{y+1}{z+1}
\]

Hence
\[
x - \frac{lb}{la} y = \frac{lb}{la} - 1
\]
\[
x - \frac{lc}{la} z = \frac{lc}{la} - 1
\]
\[
y - \frac{lc}{lb} z = \frac{lc}{lb} - 1
\]

Hence, and from equation (1) the relation required may be found.

847. Let \( y = A + Bx + Cx^2 + Dx^3 + \&c. \) be the equation to the curve. Then if \( k \) be the increment of the ordinate, and \( h \) that of the abscissa by Taylor's Theorem, we have
\[
y + k = y + \frac{dy}{dx} h + \frac{d^2y}{dx^2} \frac{h^2}{2} + \&c.
\]
\[
\therefore k = \frac{dy}{dx} h + \frac{d^2y}{dx^2} \frac{h^2}{2} + \&c.
\]

848. Let \( x \) be one of the parts. Then
\[
x^2 \cdot (30 - x) = \text{max}.
\]
\[
\therefore 2x(30 - x) = x^2
\]
and \( x = 10. \)

849. Let \( x \) be one part. Then
\[
x^2 \sqrt{100 - x} = \text{max}.
\]
\[
\therefore x^4 (100 - x) = \text{max}.
\]
and \( 4x^3 (100 - x) = x^4. \)

Hence \( x = 80. \)
850. Let $\theta$ be the arc required, of the circle whose radius is 1. Then the chord is $2 \sin \frac{\theta}{2}$, and we have, by the question,

$$\sin \frac{\theta}{2} \cdot \cos \theta = \max.$$  

$$\therefore \quad \frac{d\theta}{2} \cos \theta \times \cos \frac{\theta}{2} = d\theta \sin \theta \cdot \sin \frac{\theta}{2}$$  

$$\therefore \quad \tan \theta \cdot \tan \frac{\theta}{2} = \frac{1}{2}$$  

and

$$\frac{2 \tan^2 \frac{\theta}{2}}{1 - \tan^2 \frac{\theta}{2}} = \frac{1}{2}.$$  

Hence

$$\tan \frac{\theta}{2} = \frac{1}{\sqrt{5}}.$$  

$$\therefore \quad \tan \theta = \sqrt{5} \frac{2}{1 - 2} = \frac{\sqrt{5}}{2}$$  

which, by means of the tables, will give the arc required.

851. Since $\sqrt{\frac{a^2 + x^2}{a + x}} = \min.$

$$\therefore \quad \frac{a^2 + x^2}{a + x} = \min. = u$$  

$$\therefore \quad \frac{du}{dx} = \frac{2x}{a + x} - \frac{a^2 + x^2}{(a + x)^2} = 0$$  

$$\therefore \quad 2ax + x^2 = a^2$$  

and $x = - a \pm \sqrt{2a^2}$

$$= - a (1 \mp \sqrt{2})$$  

852. Since $x + y + z = a$ . . . . . . . . . (1)

and $x^n \cdot y^n \cdot z^n = \max.$
or \( mx + nly + rz = \text{max} \).

\[
\begin{align*}
\frac{dx}{dx} &= -z = -\frac{n}{m} \cdot x \\
\frac{dy}{dy} &= -x = -\frac{r}{n} \cdot y \\
\frac{dz}{dz} &= x = -\frac{m}{n} \cdot z
\end{align*}
\]

whence and from equat. (1) the values of \( x, y, z \), which give the maximum will be found.

853. Let \( x \) be the number. Then

\[
\frac{1}{m} x^m - \frac{1}{n} x^n = \text{max}.
\]

\[
\therefore \frac{1}{m} x^{m-1} = \frac{1}{n} x^{n-1}
\]

\[
\therefore x = \left( \frac{m}{n} \right)^{\frac{1}{m-n}}
\]

854. Let \( P = a^x - b^x \). Then

\[
\frac{dP}{dx} = a^x \ln a - b^x \ln b
\]

\[
= a^x \ln a - b^x \ln b = l \frac{a}{b} \quad \text{when} \ x = a,
\]

which is the value required.

Again, let \( P = 1 - x, Q = \cot \frac{\pi x}{2} \),

Then \( \frac{dp}{dx} = -1, \frac{dQ}{dx} = \frac{\pi}{2 \sin^2 \frac{\pi x}{2}} \)

\[
\therefore \frac{dP}{dQ} = 2 \sin^2 \frac{\pi x}{2} = \frac{2}{\pi}
\]

when \( x = 1 \).

855. Let the coefficients he denoted by \( P_2, P_4, P_6, \&c. \), \( P_2, \&c. \) Then

\[
P_2 = ma, P_4 = n \cdot \frac{n-2}{2} a^x, - P_6 = n \cdot \frac{n-4}{2} \cdot \frac{n-5}{2} a^x
\]

\&c. \&c., and \( \pm P_2 = \frac{n(n-x+1)(n-x+2)\ldots(n-2r+1)x}{2 \cdot 3 \ldots \cdot r} \).
which law of the coefficients, however, is not indicated in the enunciation of the problem.

Now we know that

\[ S_2 = P_1 S_1 - 2P_2 = -2P_2 = 2na \]

\[ S_1 = -P_2 S_2 - 4P_4 = 2n^2 a^2 - 2n(n - 3)^2 \]

\[ = 6na^2 = \frac{4 \cdot 3}{2} \times na^2 \]

\[ S_5 = -P_5 S_4 - P_4 S_3 - 6P_6 = 6n^2 a^3 - n \cdot \frac{n - 3}{2} a^2 \times 2na \]

\[ = 20na^3 = \frac{6 \times 5 \times 4}{2 \cdot 3} \times na^3 \]

&c. = &c.

856. \[
\frac{du}{d} = \frac{x - y}{x + y} \left(1 + \left(\frac{x - y}{x + y}\right)^2\right) = \frac{(dx - dy)(x + y) - (dx + dy)(x - y)}{(x + y)^2 + (x - y)^2} = \frac{2y dx - 2xdy}{2x^2 + 2y^2} = \frac{y dx - x dy}{x^2 + y^2}
\]

857. First we have

\[
\frac{x - x^2}{1 - x^2} = \frac{x}{1 + x} \times \frac{1 - x}{1 - x} = \frac{x}{1 + x}
\]

which, when \( x = 1 \), becomes

\[
\frac{1}{2}.
\]

858. Since \( \frac{dy}{dx} = \frac{a^2 + x^2 - y^2}{a^2} \), we have

\[ a^2 (dy - dx) + (y^2 - x^2) dx = 0, \]

which admits being simplified by making

\[ y - x = u; \]
for thence we get
\[ a^2 \frac{du}{dx} + 2ux + u^2 = 0. \]

Again, by making
\[ -\frac{1}{u} = v \]
differentiating and substituting, we have
\[ \frac{a^2 dv}{dx} - 2xv + 1 = 0 \]
a linear equation. Let therefore
\[ v = wz \]
then
\[ \frac{a^2 zdw}{dx} + \frac{a^2 wdz}{dx} - 2xvw + 1 = 0 \]
and since we are at liberty to make a second assumption, let
\[ \frac{a^2 zdw}{dx} - 2xw = 0 \]

\[ \begin{align*}
\frac{a^2}{2} \cdot \frac{dw}{w} &= xdx = 0 \\
and \quad a^2 wdz + dx &= 0
\end{align*} \]
the former of which gives
\[ x^2 = a^2 lw + \text{const.} = a^2 lw \]
\[ \therefore w = \frac{1}{c} e^{\frac{x^2}{a^2}} \]

Hence
\[ dz = -\frac{dx}{a^2 w} = -\frac{c}{a^2} \cdot e^{-\frac{x^2}{a^2}} dx \]
whose integral has never yet been found except between certain limits. See Laplace Mec. Cel. liv. X. art 5., and Whewell's Dynamics pp. 15 and 16.

END OF VOL. I.